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SID 62-387

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MATHEMATICAL OUTLINES

FOR DESIGN OF APOLLO

CREW TRAINING EQUIPMENT [U]

4.5.2.14

NAS 9-150
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Prepared by

TRAINING SYSTEMS
REQUIREMENTS

APOLLO - GSE

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Technical Information
Center
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NORTH AMERICAN AVIATION, INC.
SPACE and INFORMATION SYSTEMS DIVISION

FOREWORD

In the development of training equipment designed to support the training requirements of the Apollo program, a preliminary study of the various spacecraft flight problems has been made. These studies involve the complete lunar mission and include lunar environment studies as well as earth environment studies.

Through the use of ordinary dynamics and mathematical techniques, an effort has been made to describe the flight of the Apollo vehicle from launch to lunar landing and return. This description is intended to provide a criteria for analog and digital computer design.

The descriptions included, utilize state of the art simulation standard expressions for flying vehicles. Much of the data has been derived from formerly proven techniques in mathematical description, which have been used for computer design criteria. Other data included has been obtained from NAA preliminary Apollo vehicle analog computer studies.

The environment descriptions are both lunar and earth. Mathematical descriptions are devised to provide a non-polar orbit and a polar orbit. Selection of these descriptions depend upon the necessity of using polar orbit or non-polar orbit. There are differences in complexity of the computing equipment, and also selection can be made between analog and digital requirements, in establishing polar and non-polar mathematical descriptions.

The lunar description has been devised using dynamic analysis and celestial mechanics data. Information inputs have been from many sources to devise the lunar description. At this writing the lunar mathematical formulation is considered basically sound; however, the lunar study is not complete.

The formulations provided are suitable for digital and hybrid analog/ digital computation. With ordinary hybrid digital and analog computing equipment, the translation equations would be solved digitally while the Euler equations would be solved by analog methods. For a completely digital computation, it is suggested that the Euler Angle equations be solved utilizing quaternion description of the nine Euler Angle transformations. It is expected that computations would be excessive for a completely analog computation for any of the models presented.

These models presented have the desirable feature that the coordinates used in the translation equations do not depend upon the vehicles angular orientation. As a result, the components of the vehicles velocity in these coordinate systems cannot change value rapidly unless the velocity vector of the vehicle changes magnitude or direction rapidly. This should serve to allow improved computer dynamic characteristics. To further improve computer characteristics, the "gimbal lock" of the Euler transformation has been removed from the pitch axis and inserted in the roll axis. This will allow tumbling of the Apollo vehicle in the pitch plane. This should provide better overall simulation of the vehicle flight.

These formulations are expected to provide the basic computer descriptions within the development of all Part Task Trainers and the Mission Simulator.

NOMENCLATURE

$\bar{a}$	ACCELERATION VECTOR OF VEHICLE RELATIVE TO INERTIAL AXES.
f	PARAMETER DESCRIBING EARTH OBLATENESS.
$\bar{F}$	AERODYNAMIC FORCE VECTOR.
F	WITH APPROPRIATE SUBSCRIPTS, COMPONENTS OF $\bar{F}$ REFERRED TO SELECTED AXES.
$\bar{G}$	GRAVITY FORCE VECTOR.
h	ALTITUDE.
$\bar{H}$	CONTROL FORCE VECTOR.
H	WITH APPROPRIATE SUBSCRIPTS, COMPONENTS OF $\bar{H}$ REFERRED TO SELECTED AXES.
$\bar{i}, \bar{j}, \bar{k},$	UNIT VECTORS ALONG THE X, Y, Z AXES, RESPECTIVELY; WITH APPROPRIATE SUBSCRIPTS, UNIT VECTORS ALONG AXES IN OTHER SETS.
I_{xx}, I_{yy}, I_{zz}	MOMENTS OF INERTIA OF VEHICLE ABOUT THE X_B, Y_B, Z_B AXES, RESPECTIVELY.
I_{xz}	PRODUCT OF INERTIA OF VEHICLE ABOUT X_B AND Z_B AXES.
J	WITH APPROPRIATE SUBSCRIPTS, COMPONENTS OF CONTROL MOMENT.
K	GRAVITY CONSTANT
L, M, N	COMPONENTS OF THE AERODYNAMIC MOMENT, REFERRED TO THE X_B, Y_B, Z_B AXES, RESPECTIVELY.
m	VEHICLE MASS.
$\bar{P}$	PROPULSIVE FORCE VECTOR.
P	WITH APPROPRIATE SUBSCRIPTS, COMPONENTS OF $\bar{P}$ REFERRED TO SELECTED AXES.
P, Q, R	COMPONENTS OF ANGULAR VELOCITY OF VEHICLE RELATIVE TO INERTIAL AXES, REFERRED TO THE X_B, Y_B, Z_B AXES RESPECTIVELY.
$\bar{r}$	RADIUS VECTOR FROM EARTH CENTER TO VEHICLE CENTROID.
r	LENGTH OF $\bar{r}$.
δr	DIFFERENCE BETWEEN r AND R_0.
R_0	EARTH RADIUS IN THE EQUATORIAL PLANE.
R_e	EARTH RADIUS AT A LOCAL POINT ON THE EARTH'S SURFACE.
R_1	ARITHMETIC MEAN OF THE EARTH'S RADIUS AT POLE AND EQUATOR.

|	
t	TIME.
T	WITH APPROPRIATE SUBSCRIPTS, COMPONENTS OF PROPULSIVE MOMENT.
∇	VELOCITY VECTOR OF VEHICLE RELATIVE TO INERTIAL AXES.
∇_a	VELOCITY VECTOR OF VEHICLE RELATIVE TO THE AIR
V_a	WITH APPROPRIATE SUBSCRIPTS, COMPONENTS OF ∇_a REFERRED TO SELECTED AXES.
∇_E	VELOCITY VECTOR OF VEHICLE RELATIVE TO THE X_E, Y_E, Z_E FRAME.
∇_W	WIND VELOCITY VECTOR RELATIVE TO THE X_E, Y_E, Z_E FRAME.
W_W	WITH APPROPRIATE SUBSCRIPTS, COMPONENTS OF ∇_W REFERRED TO SELECTED AXES.
V_a	MAGNITUDE OF ∇_a.
X, Y, Z	INERTIAL AXES. WITH APPROPRIATE SUBSCRIPTS, OTHER AXIS SYSTEMS.
α	ANGLE OF ATTACK.
β	ANGLE OF SIDESLIP.
θ	EULER ANGLE ESTABLISHING VEHICLE ORIENTATION.
λ	ANGLE ESTABLISHING INITIAL DIRECTION OF NOMINAL TRAJECTORY.
λ_F	ANGLE ESTABLISHING LOCAL DIRECTION OF NOMINAL TRAJECTORY.
μ	GRAVITY CONSTANT.
ϕ	GEOCENTRIC LATITUDE OF VEHICLE.
ϕ_i	GEOCENTRIC LATITUDE OF LAUNCH POINT.
ϕ_F	SPHERICAL ANGULAR COORDINATE OF VEHICLE MEASURED NORMAL TO NOMINAL TRAJECTORY PLANE.
ϕ	EULER ANGLE ESTABLISHING VEHICLE ORIENTATION.
Ψ	GEOCENTRIC LONGITUDE OF VEHICLE MEASURED FROM LAUNCH POINT.
Ψ_F	SPHERICAL ANGULAR COORDINATE OF VEHICLE MEASURED IN NOMINAL TRAJECTORY PLANE.
ψ	EULER ANGLE ESTABLISHING VEHICLE ORIENTATION.
Ω_E	EARTH ROTATIONAL VELOCITY.
$\overline{\omega}_B$	ANGULAR VELOCITY VECTOR OF VEHICLE RELATIVE TO INERTIAL FRAME.
$\overline{\omega}_E$	ANGULAR VELOCITY VECTOR OF X_E, Y_E, Z_E FRAME RELATIVE TO INERTIAL FRAME.

$\overline{\omega}_B$ ANGULAR VELOCITY VECTOR OF X_B, Y_B, Z_B FRAME RELATIVE TO INERTIAL FRAME.

$\overline{\omega}_V$ ANGULAR VELOCITY VECTOR OF X_V, Y_V, Z_V FRAME RELATIVE TO INERTIAL FRAME.

$\overline{\omega}_F$ ANGULAR VELOCITY VECTOR OF X_F, Y_F, Z_F FRAME RELATIVE TO INERTIAL FRAME.

$\overline{\omega}_{FE}$ ANGULAR VELOCITY VECTOR OF X_F, Y_F, Z_F FRAME RELATIVE TO X_E, Y_E, Z_E FRAME.

SUBSCRIPTS

B BODY AXES.

E EARTH ROTATING AXES.

F EARTH-VEHICLE GEOCENTRIC AXES REFERRED TO NOMINAL TRAJECTORY PLANE.

G EARTH-VEHICLE GEOCENTRIC AXES REFERRED TO EQUATORIAL PLANE.

V VEHICLE GEOCENTRIC AXES.

W VEHICLE WIND AXES.

x, y, z COMPONENTS ALONG BODY AXES X_B, Y_B, Z_B RESPECTIVELY.

r, θ, ϕ COMPONENTS ALONG X_G, Y_G, Z_G AXES RESPECTIVELY.

r, θ_F, ϕ_F COMPONENTS ALONG X_F, Y_F, Z_F AXES RESPECTIVELY.

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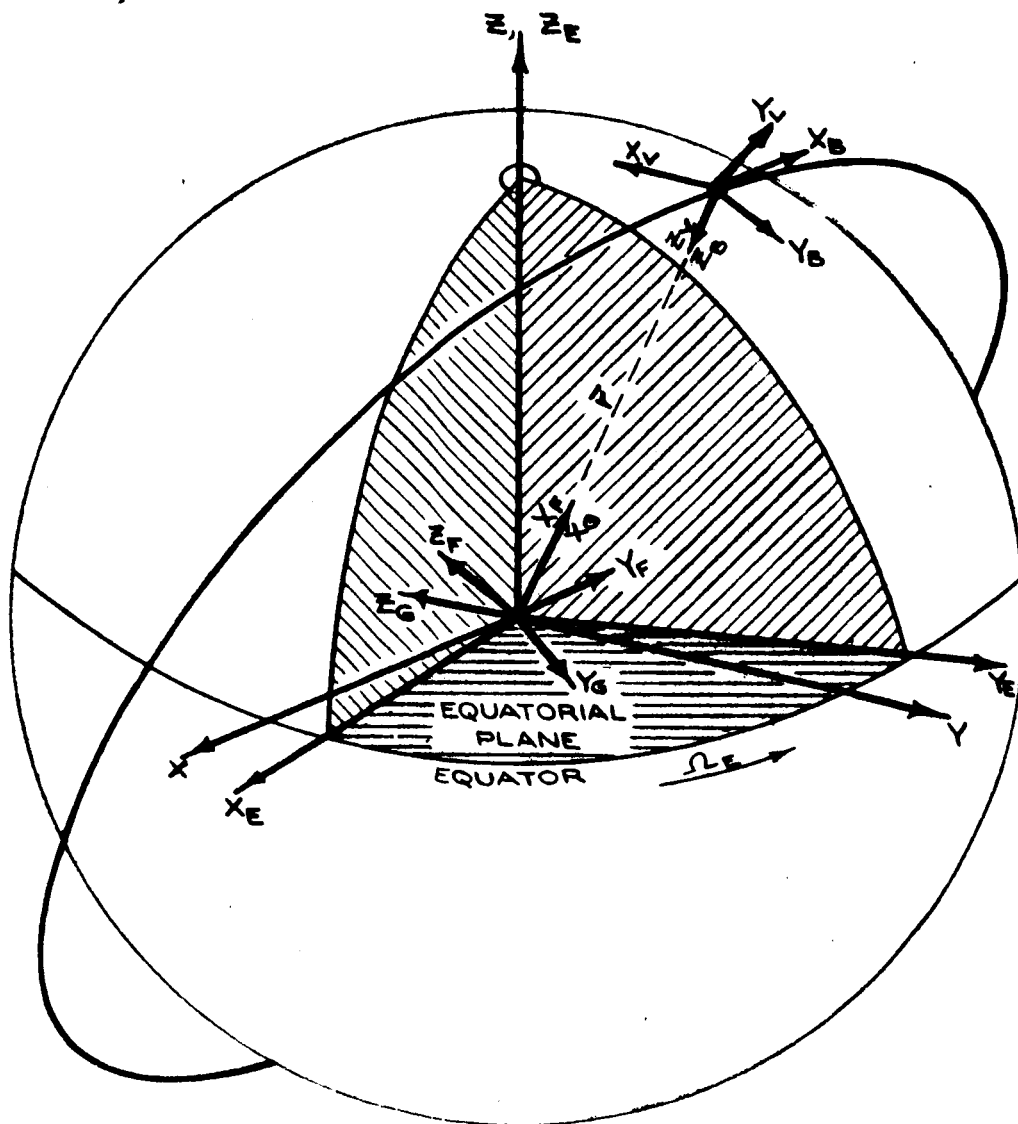
NOTE: WIND AXES, (X_W, Y_W, Z_W), NOT SHOWN.

Figure 1

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I   INERTIAL AXES (X, Y, Z)

1. ORIGIN AT EARTH CENTER.
2. UNIT VECTORS (i, j, k).
3. Z-AXIS COINCIDENT WITH EARTH POLAR AXIS, POSITIVE NORTH.
4. $X - Z$ PLANE CONTAINS INITIAL POSITION OF VEHICLE.

II   EARTH AXES (X_E, Y_E, Z_E)

1. ORIGIN AT EARTH CENTER.
2. UNIT VECTORS (i_E, j_E, k_E).
3. Z_E-AXIS COINCIDENT WITH EARTH POLAR AXIS, POSITIVE NORTH.
4. INITIAL POSITION COINCIDENT WITH INERTIAL AXES.

III   EARTH-VEHICLE ORBIT PLANE GEOCENTRIC AXES (X_F, Y_F, Z_F)

1. ORIGIN AT EARTH CENTER.
2. X_F-AXIS PASSES THROUGH VEHICLE CENTROID.
3. $X_F - Y_F$ PLANE IS NOMINAL TRAJECTORY PLANE.
4. Y_F POINTS ESSENTIALLY IN DIRECTION OF FLIGHT.
5. Z_F POINTS LEFT WHEN LOOKING IN DIRECTION OF FLIGHT.
6. UNIT VECTORS (i_F, j_F, k_F).

IV   EARTH-VEHICLE GEOCENTRIC AXES (X_G, Y_G, Z_G)

1. ORIGIN AT EARTH CENTER.
2. X_G-AXIS PASSES THROUGH VEHICLE CENTROID.
3. $X_G - Z_G$ PLANE CONTAINS EARTH POLAR AXIS.
4. Y_G-AXIS LIES IN THE EQUATORIAL PLANE.
5. Z_G-AXIS IS POSITIVE NORTH OF EQUATORIAL PLANE.
6. UNIT VECTORS (i_G, j_G, k_G).

V   VEHICLE BODY AXES (X_B, Y_B, Z_B).

1. ORIGIN AT VEHICLE CENTROID.
2. $X_B - Z_B$ PLANE COINCIDENT WITH PLANE OF SYMMETRY OF VEHICLE.
3. Z_B-AXIS POSITIVE DOWNWARD, NORMAL TO X_B-AXIS.
4. X_B-AXIS POSITIVE FORWARD.
5. Y_B-AXIS POSITIVE RIGHT LOOKING FORWARD AND NORMAL TO X_B.
6. UNIT VECTORS (i_B, j_B, k_B).

VI VEHICLE WIND AXES (X_w, Y_w, Z_w).

1. ORIGIN AT VEHICLE CENTROID.
2. X_w-AXIS POINTS IN DIRECTION OF VEHICLE VELOCITY RELATIVE TO AIR.
3. Z_w-AXIS LIES IN PLANE OF SYMMETRY OF VEHICLE.
4. Y_w-AXIS POSITIVE RIGHT LOOKING TOWARD POSITIVE X_w AND NORMAL TO X_w.
5. THESE AXES ARE REACHED FROM X_B, Y_B, Z_B AXES BY ROTATION.
   - a) ABOUT Y_B-AXIS THROUGH ANGLE $(-\alpha)$.
   - b) ABOUT Z_B-AXIS THROUGH ANGLE (β).
6. UNIT VECTORS (i_w, j_w, k_w).

VII VEHICLE GEOCENTRIC AXES (X_v, Y_v, Z_v)

1. ORIGIN AT VEHICLE CENTROID.
2. X_v-AXIS POSITIVE NORTH.
3. Y_v-AXIS POSITIVE EAST.
4. Z_v-AXIS PASSES THROUGH EARTH CENTER.
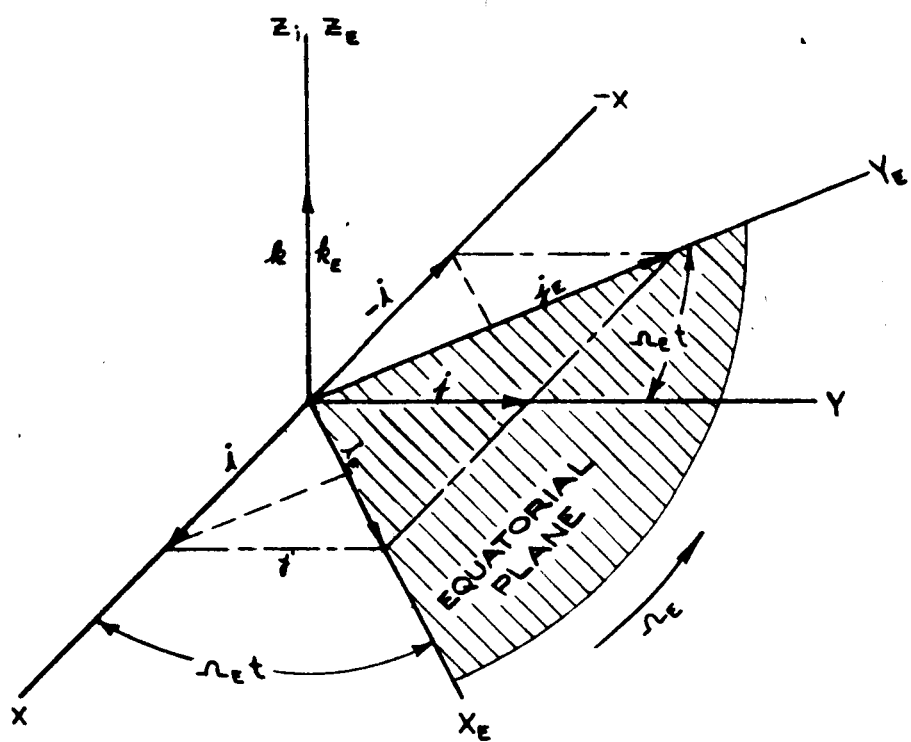5. X_v-Z_v PLANE CONTAINS EARTH POLAR AXIS.
6. UNIT VECTORS (i_v, j_v, k_v).

TRANSFORMATION FROM INERTIAL TO EARTH AXES

INERTIAL AXES: X, Y, Z
EARTH AXES: X_E, Y_E, Z_E,

UNIT VECTORS: (i, j, k) AND (i_E, j_E, k_E).

$$i_E = i \cos \Omega_E t + j \sin \Omega_E t + k(0)$$

$$j_E = -i \sin \Omega_E t + j \cos \Omega_E t + k(0)$$

$$k_E = i(0) \qquad + j(0) \qquad + k(1)$$

$$\begin{Bmatrix} i_E \\ j_E \\ k_E \end{Bmatrix} = \begin{bmatrix} \cos \Omega_E t & \sin \Omega_E t & 0 \\ -\sin \Omega_E t & \cos \Omega_E t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} i \\ j \\ k \end{Bmatrix}$$
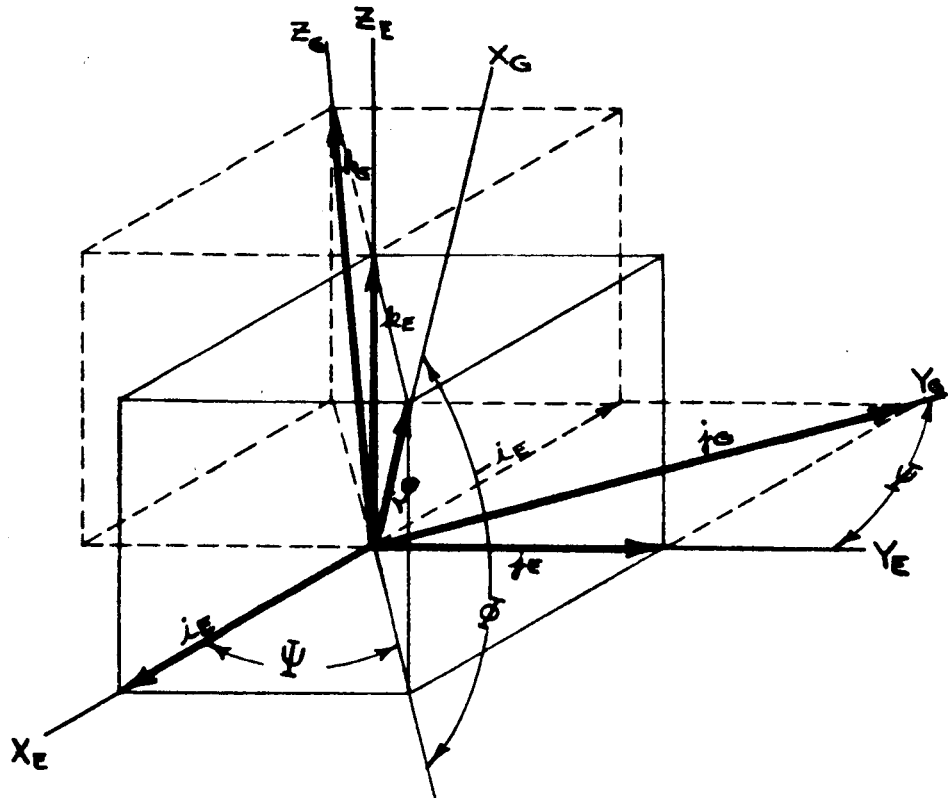
$[1]$

Figure 2

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<u>ROTATION FROM EARTH AXES TO EARTH-VEHICLE GEOCENTRIC AXES</u>

<u>EARTH ROTATING AXES (X_E, Y_E, Z_E).</u>

<u>EARTH-VEHICLE GEOCENTRIC AXES (X_G, Y_G, Z_G).</u>

<u>UNIT VECTORS : (i_E, j_E, k_E) AND (i_G, j_G, k_G).</u>

$$\underline{i_G = (i_E \cos \Psi + j_E \sin \Psi) \cos \Phi + k_E \sin \Phi}$$

$$\underline{j_G = -i_E \sin \Psi + j_E \cos \Psi + k_E(0)}$$

$$\underline{k_G = (-i_E \cos \Psi - j_E \sin \Psi) \sin \Phi + k_E \cos \Phi}$$

$$
\begin{Bmatrix} i_G \\ j_G \\ k_G \end{Bmatrix}
=
\begin{bmatrix}
\cos \Phi & 0 & \sin \Phi \\
0 & 1 & 0 \\
-\sin \Phi & 0 & \cos \Phi
\end{bmatrix}
\underbrace{\qquad}_{[3]}
\begin{bmatrix}
\cos \Psi & \sin \Psi & 0 \\
-\sin \Psi & \cos \Psi & 0 \\
0 & 0 & 1
\end{bmatrix}
\underbrace{\qquad}_{[2]}
\begin{Bmatrix} i_E \\ j_E \\ k_E \end{Bmatrix}
$$

Figure 3

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ROTATION FROM EARTH-VEHICLE GEOCENTRIC AXES TO VEHICLE GEOCENTRIC AXES
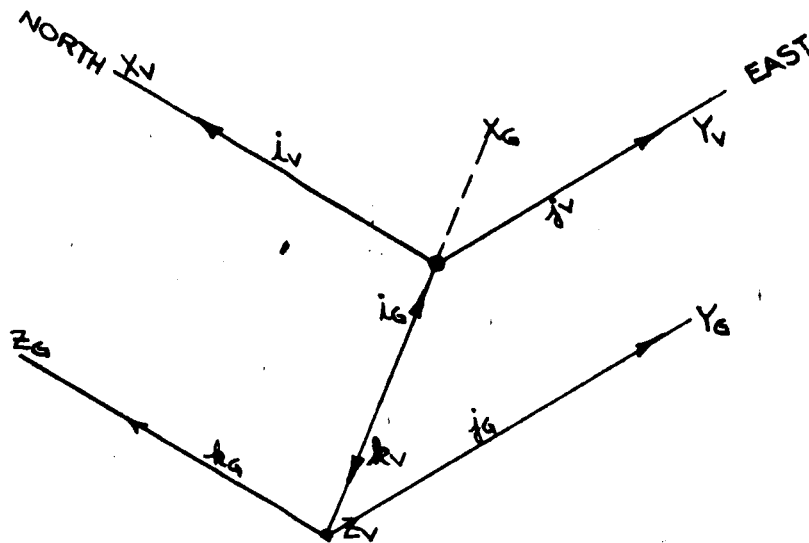
EARTH-VEHICLE GEOCENTRIC AXES (X_G, Y_G, Z_G)

VEHICLE GEOCENTRIC AXES (X_V, Y_V, Z_V)

UNIT VECTORS: (i_G, j_G, k_G) AND (i_V, j_V, k_V)

$X_G - Z_G$ PLANE CONTAINS EARTH POLAR AXIS

$X_V - Z_V$ PLANE CONTAINS EARTH POLAR AXIS

$$i_V = i_G(0) + j_G(0) + k_G(1)$$

$$j_V = i_G(0) + j_G(1) + k_G(0)$$

$$k_V = i_G(-1) + j_G(0) + k_G(0)$$

$$\begin{Bmatrix} i_V \\ j_V \\ k_V \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} i_G \\ j_G \\ k_G \end{Bmatrix}$$
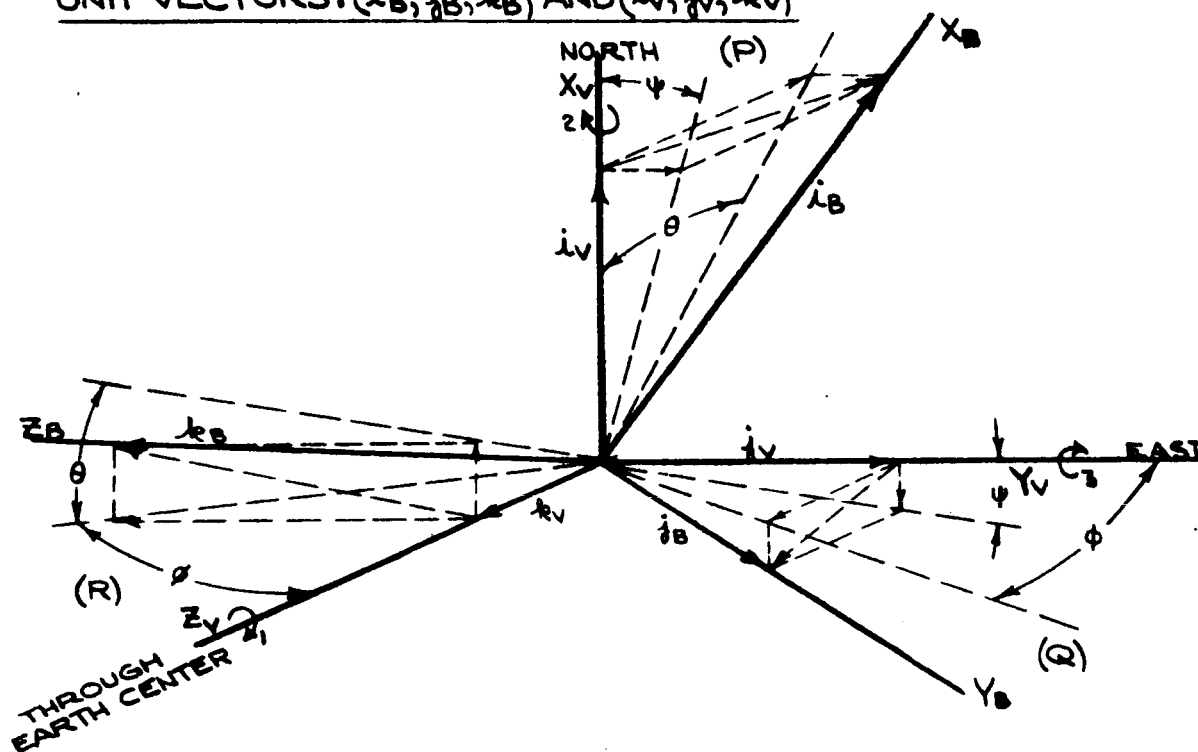
$$[\text{\textcircled{4}}]$$

Figure 4

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ORIENTATION OF VEHICLE BODY AXES RELATIVE TO THE

VEHICLE GEOCENTRIC AXES

VEHICLE BODY AXES (X_B, Y_B, Z_B)

VEHICLE GEOCENTRIC AXES (X_V, Y_V, Z_V)

UNIT VECTORS: (i_B, j_B, k_B) AND (i_V, j_V, k_V)

$$\begin{Bmatrix} i_B \\ j_B \\ k_B \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}}_{[7]} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}}_{[6]} \underbrace{\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[5]} \begin{Bmatrix} i_V \\ j_V \\ k_V \end{Bmatrix}$$

Figure 5

ORIENTATION OF VEHICLE BODY AXES RELATIVE TO THE

VEHICLE GEOCENTRIC AXES

THE ORIENTATION OF VEHICLE BODY AXES RELATIVE TO THE VEHICLE GEOCENTRIC AXES, AS SHOWN IN FIGURE 5, IS ACCOMPLISHED IN THREE STEPS AS FOLLOWS:

STEP NO.1: ROTATION ABOUT THE Z_B-AXIS THROUGH ANGLE ψ AS SHOWN IN FIGURE 5A, WHERE (B") DENOTES INTERMEDIATE POSITION OF X_B, Y_B, Z_B.
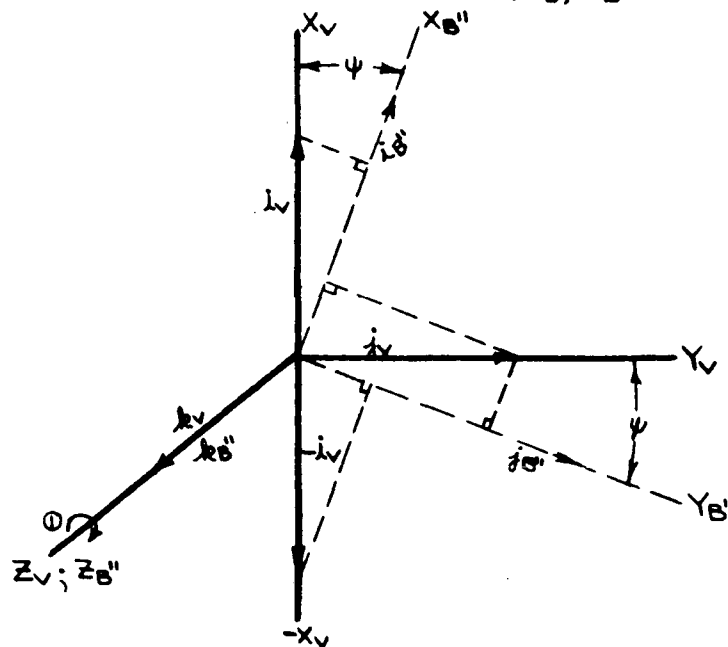

FIGURE 5A

$$\mathbf{i}_{B''} = \mathbf{i}_V \cos\psi + \mathbf{j}_V \sin\psi + \mathbf{k}_V (0)$$
$$\mathbf{j}_{B''} = -\mathbf{i}_V \sin\psi + \mathbf{j}_V \cos\psi + \mathbf{k}_V (0)$$
$$\mathbf{k}_{B''} = \mathbf{i}_V (0) + \mathbf{j}_V (0) + \mathbf{k}_V (1)$$

$$\begin{Bmatrix} \mathbf{i}_{B''} \\ \mathbf{j}_{B''} \\ \mathbf{k}_{B''} \end{Bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{i}_V \\ \mathbf{j}_V \\ \mathbf{k}_V \end{Bmatrix}$$

$$[5]$$

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STEP NO. 2: ROTATION ABOUT THE X_B'-AXIS THROUGH
ANGLE ϕ AS SHOWN IN FIGURE 5B, WHERE
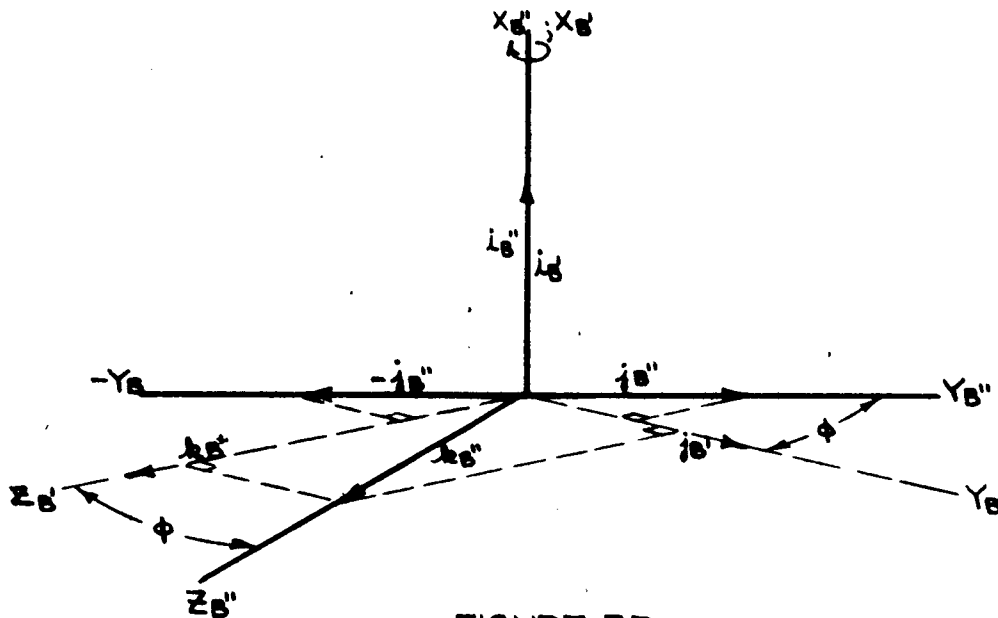(B')DENOTES INTERMEDIATE POSITION OF X_B, Y_B, Z_B.

FIGURE 5B

$$i_{B'} = i_{B''}(1) + j_{B''}(0) + k_{B''}(0)$$

$$j_{B'} = i_{B''}(0) + j_{B''}\cos\phi + k_{B''}\sin\phi$$

$$k_{B'} = i_{B''}(0) - j_{B''}\sin\phi + k_{B''}\cos\phi$$

$$\begin{Bmatrix} i_{B'} \\ j_{B'} \\ k_{B'} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} i_{B''} \\ j_{B''} \\ k_{B''} \end{Bmatrix}$$

$$[\circled{6}]$$

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STEP NO. 3: ROTATION ABOUT THE Y_B-AXIS THROUGH ANGLE θ AS SHOWN IN FIGURE 5C.
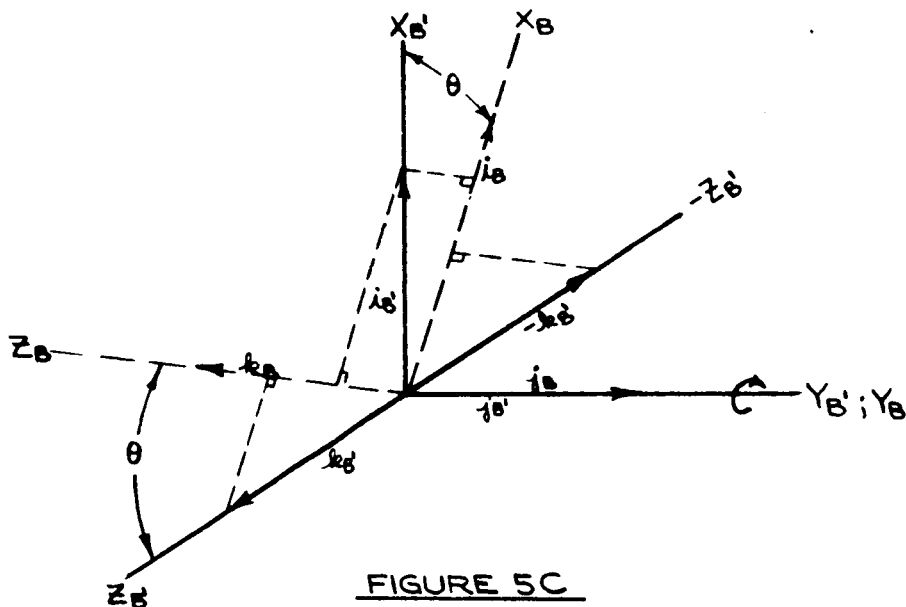

FIGURE 5C

$$i_B = i_{B'}\cos\theta + j_{B'}(0) - k_{B'}\sin\theta$$

$$j_B = i_{B'}(0) + j_{B'}(1) + k_{B'}(0)$$

$$k_B = i_{B'}\sin\theta + j_{B'}(0) + k_{B'}\cos\theta$$

$$
\begin{Bmatrix} i_B \\ j_B \\ k_B \end{Bmatrix} =
\underbrace{\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}}
\begin{Bmatrix} i_{B'} \\ j_{B'} \\ k_{B'} \end{Bmatrix}
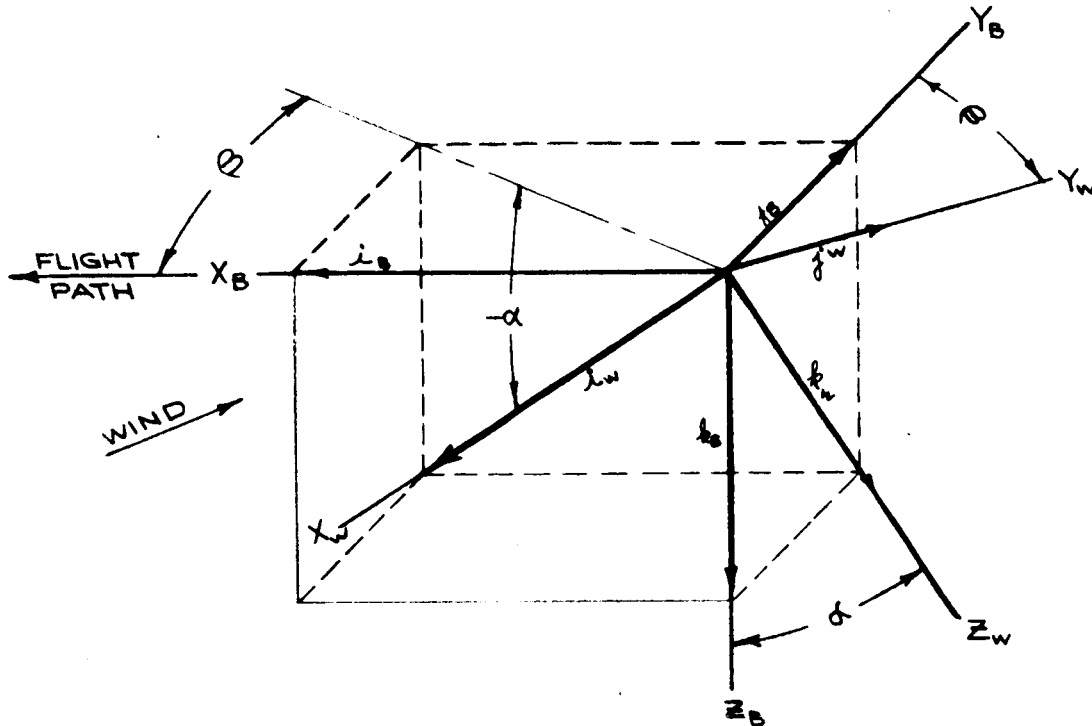$$

$$[\text{⑦}]$$

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VEHICLE BODY-WIND AXES

VEHICLE BODY AXES (X_B, Y_B, Z_B).

VEHICLE WIND AXES (X_W, Y_W, Z_W).

UNIT VECTORS: (i_B, j_B, k_B) AND (i_W, j_W, k_W).

$$i_w = (i_B \cos \alpha + k_B \sin \alpha) \cos \beta + j_B \sin \beta$$

$$j_w = j_B \cos \beta - (i_B \cos \alpha + k_B \sin \alpha) \sin \beta$$

$$k_w = -i_B \sin \alpha + k_B \cos \alpha$$

$$
\begin{Bmatrix} i_w \\ j_w \\ k_w \end{Bmatrix} =
\begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}
\underbrace{}_{[9]}
\begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}
\underbrace{}_{[8]}
\begin{Bmatrix} i_B \\ j_B \\ k_B \end{Bmatrix}
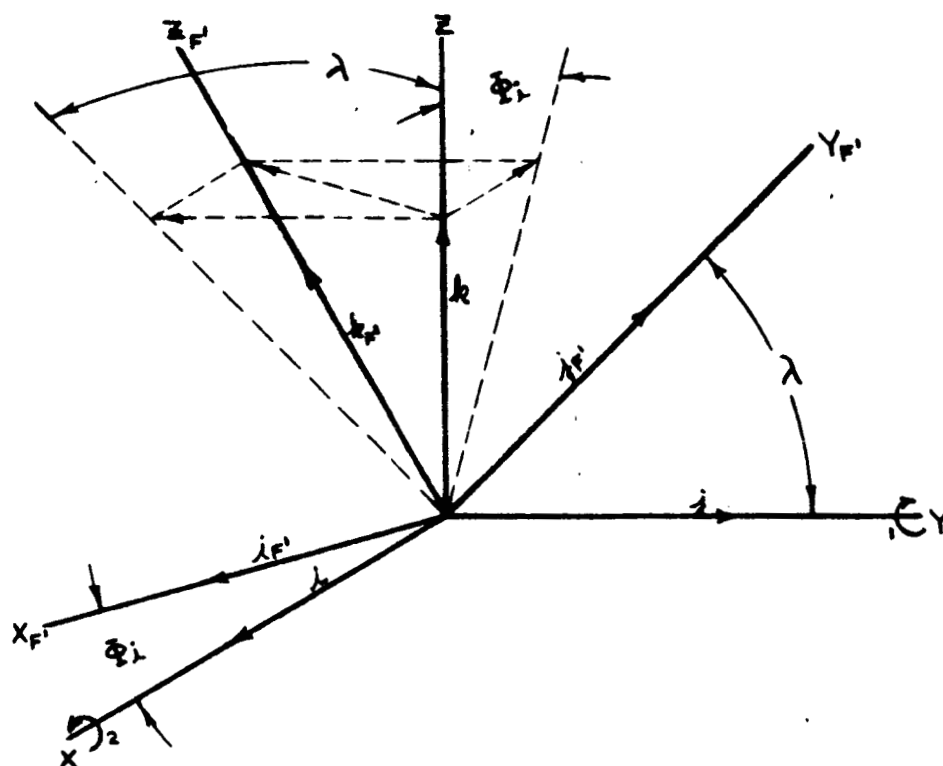$$

Figure 6

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ORIENTATION OF ORBIT PLANE RELATIVE TO INERTIAL AXES

INERTIAL AXES (X, Y, Z)

X_E', Y_E', Z_E' — INTERMEDIATE POSITION OF X_F, Y_F, Z_F.

UNIT VECTORS: (i, j, k) AND (i_F, j_F', k_F')

Φ_i — GEOCENTRIC LATITUDE OF LAUNCH POINT.

λ — ANGLE ESTABLISHING INITIAL DIRECTION OF
　　 NOMINAL TRAJECTORY.

Figure 7A

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ORIENTATION OF ORBIT PLANE RELATIVE TO INERTIAL AXES

EARTH-VEHICLE ORBIT PLANE GEOCENTRIC AXES (X_F, Y_F, Z_F).

X_F', Y_F', Z_F' - INTERMEDIATE POSITION OF X_F, Y_F, Z_F.

UNIT VECTORS: (i_F, j_F, k_F) AND $(i_{F'}, j_{F'}, k_{F'})$.
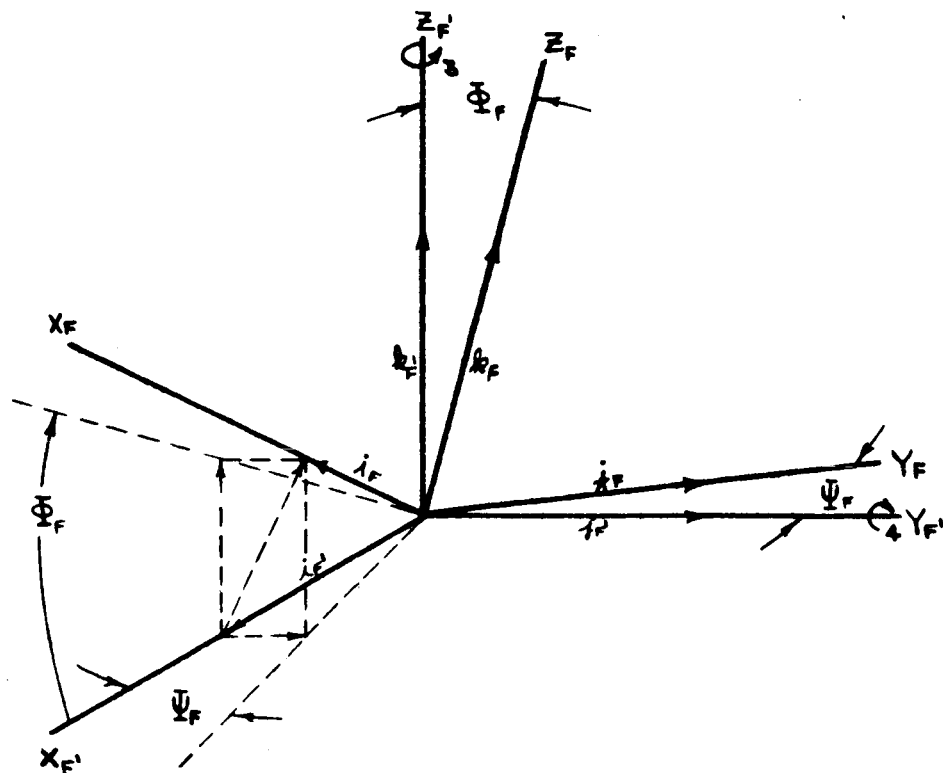
Ψ_F - SPHERICAL ANGULAR COORDINATE OF VEHICLE
MEASURED IN NOMINAL TRAJECTORY PLANE, ABOUT
Z-AXIS.

Φ_F - SPHERICAL ANGULAR COORDINATE OF VEHICLE
MEASURED NORMAL TO NOMINAL TRAJECTORY,
ABOUT Y-AXIS.

Figure 7B

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ORIENTATION OF ORBIT PLANE RELATIVE TO INERTIAL AXES

THE ORIENTATION OF ORBIT PLANE RELATIVE TO
INERTIAL AXES IS ACCOMPLISHED IN FOUR STEPS AS
SHOWN IN FIGURE 7A AND FIGURE 7B AND IS AS FOLLOWS:

STEP NO.1 : ROTATION ABOUT THE Y-AXIS THROUGH ANGLE Φ_i
AS SHOWN IN FIGURE 8A.
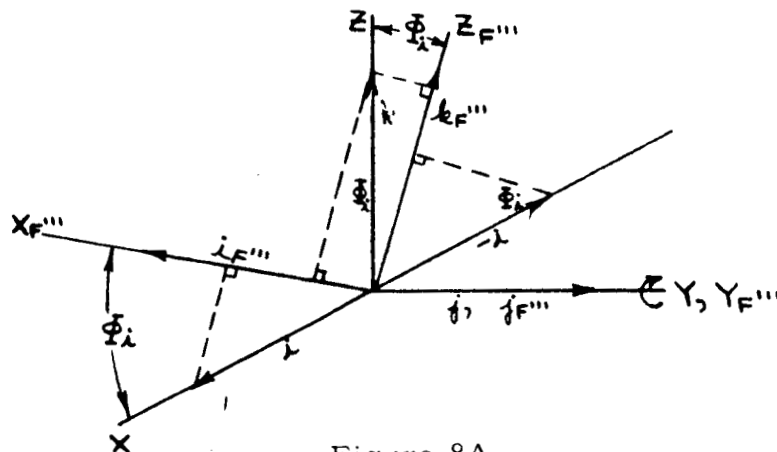
Figure 8A
(Reference Figure 7A)

COMPUTING FOR $i_{F'''}$, $j_{F'''}$, $k_{F'''}$ WE OBTAIN THE FOLLOWING:

$$i_{F'''} = i \cos \Phi_i + k \sin \Phi_i$$

$$j_{F'''} = j$$

$$k_{F'''} = i(-\sin \Phi_i) + k \cos \Phi_i$$

$$= \begin{Bmatrix} i_{F'''} \\ j_{F'''} \\ k_{F'''} \end{Bmatrix} = \begin{bmatrix} \cos \Phi_i & 0 & \sin \Phi_i \\ 0 & 1 & 0 \\ -\sin \Phi_i & 0 & \cos \Phi_i \end{bmatrix} \begin{Bmatrix} i \\ j \\ k \end{Bmatrix}$$

[10]

STEP NO.2 : ROTATION ABOUT THE $X_{F'''}$-AXIS THROUGH ANGLE λ
AS SHOWN IN FIGURE 8B.

Figure 8B
(Reference Figure 7A)

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COMPUTING FOR i_F, j_F'', k_F'' WE OBTAIN THE FOLLOWING:

$$i_F'' = i_F'''$$
$$j_F'' = j_F''' \cos \lambda + k_F''' \sin \lambda$$
$$k_F'' = -j_F''' \sin \lambda + k_F''' \cos \lambda$$

$$\begin{Bmatrix} i_F'' \\ j_F'' \\ k_F'' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda & \sin\lambda \\ 0 & -\sin\lambda & \cos\lambda \end{bmatrix} \begin{Bmatrix} i_F''' \\ j_F''' \\ k_F''' \end{Bmatrix}$$

(11)

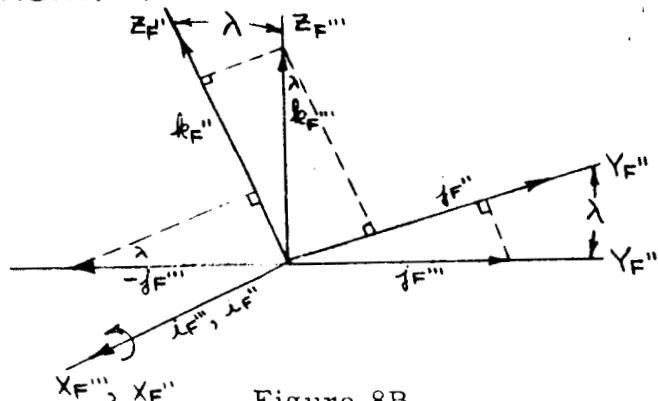<u>STEP NO.3:</u> ROTATION ABOUT THE Z_F''-AXIS THROUGH ANGLE Ψ_F AS SHOWN IN FIGURE 8C.

FIGURE 8C
(REFERENCE FIGURE 7B)

COMPUTING FOR i_F', j_F', k_F' WE OBTAIN THE FOLLOWING:

$$i_F' = i_F'' \cos \Psi_F + j_F'' \sin \Psi_F$$
$$j_F' = j_F'' \cos \Psi_F - i_F'' \sin \Psi_F$$
$$k_F' = k_F''$$

$$\begin{Bmatrix} i_F' \\ j_F' \\ k_F' \end{Bmatrix} = \begin{bmatrix} \cos\Psi_F & \sin\Psi_F & 0 \\ -\sin\Psi_F & \cos\Psi_F & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} i_F'' \\ j_F'' \\ k_F'' \end{Bmatrix}$$

(12)

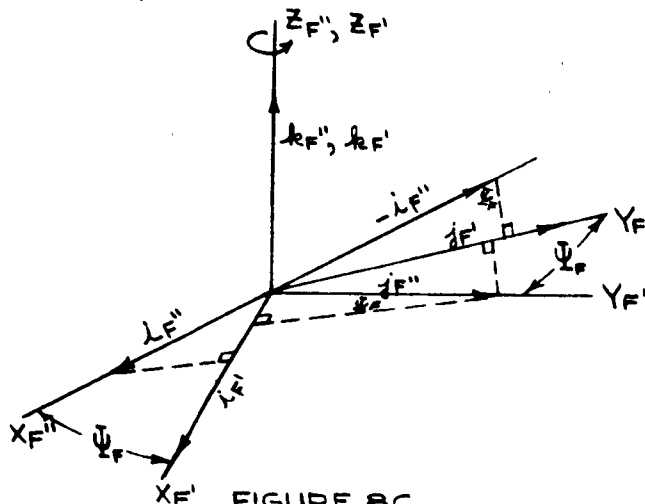STEP NO.4: ROTATION ABOUT THE Y_F'-AXIS THROUGH ANGLE Φ_F AS SHOWN IN FIGURE 8D.
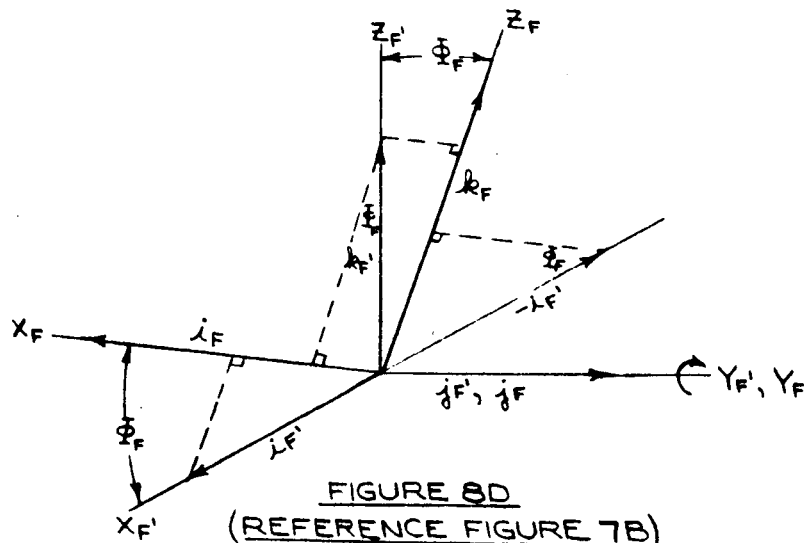
FIGURE 8D
(REFERENCE FIGURE 7B)

COMPUTING FOR i_F, j_F, k_F WE OBTAIN THE FOLLOWING:

$$i_F = i_{F'} \cos \Phi_F + k_{F'} \sin \Phi_F$$

$$j_F = j_{F'}$$

$$k_F = k_{F'} \cos \Phi_F - i_{F'} \sin \Phi_F$$

$$\begin{Bmatrix} i_F \\ j_F \\ k_F \end{Bmatrix} = \begin{bmatrix} \cos \Phi_F & 0 & \sin \Phi_F \\ 0 & 1 & 0 \\ -\sin \Phi_F & 0 & \cos \Phi_F \end{bmatrix} \begin{Bmatrix} i_{F'} \\ j_{F'} \\ k_{F'} \end{Bmatrix}$$

⑬

FINALLY BY SUBSTITUTION IN STEP NO.1 THROUGH STEP NO.4
WE OBTAIN THE FOLLOWING RESULTS:

STEP NO.1

$$i_F''' = i \cos \Phi_i + k \sin \Phi_i$$

$$j_F''' = j$$

$$k_F''' = i(-\sin \Phi_i) + k \cos \Phi_i$$

STEP NO.2

$$i_F'' = i \cos \Phi_i + k \sin \Phi_i$$

$$j_F'' = j \cos \lambda + \left[i(-\sin \Phi_i) + k \cos \Phi_i \right] \sin \lambda$$

$$k_F'' = -j \sin \lambda + \left[i(-\sin \Phi_i) + k \cos \Phi_i \right] \cos \lambda$$

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STEP NO. 5
CONVERSION OF STEP NO. 4 INTO MATRIX FORM

$$\begin{Bmatrix} i_E \\ j_E \\ k_E \end{Bmatrix} = \begin{bmatrix}
\begin{matrix}\cos\Phi_i\cos\Psi_F\cos\Phi_F\\ -\sin\Phi_i\sin\lambda\sin\Psi_F\cos\Phi_F\\ -\sin\Phi_i\cos\lambda\sin\Phi_F\end{matrix} &
\begin{matrix}\cos\lambda\sin\Psi_F\cos\Phi_F\\ -\sin\lambda\sin\Phi_F\end{matrix} &
\begin{matrix}\sin\Phi_i\cos\Psi_F\cos\Phi_F\\ +\cos\Phi_i\sin\lambda\sin\Psi_F\cos\Phi_F\\ +\cos\Phi_i\cos\lambda\sin\Phi_F\end{matrix} \\[2em]
\begin{matrix}-\sin\Phi_i\sin\lambda\cos\Psi_F\\ -\cos\Phi_i\sin\Psi_F\end{matrix} &
\cos\lambda\cos\Psi_F &
\begin{matrix}\cos\Phi_i\sin\lambda\cos\Psi_F\\ -\sin\Phi_i\sin\Psi_F\end{matrix} \\[2em]
\begin{matrix}-\sin\Phi_i\cos\lambda\cos\Phi_F\\ -\cos\Phi_i\cos\Psi_F\sin\Phi_F\\ +\sin\Phi_i\sin\lambda\sin\Psi_F\sin\Phi_F\end{matrix} &
\begin{matrix}-\sin\lambda\cos\Phi_F\\ -\cos\lambda\sin\Psi_F\sin\Phi_F\end{matrix} &
\begin{matrix}\cos\Phi_i\cos\lambda\cos\Phi_F\\ -\sin\Phi_i\cos\Psi_F\sin\Phi_F\\ -\cos\Phi_i\sin\lambda\sin\Psi_F\sin\Phi_F\end{matrix}
\end{bmatrix} \begin{Bmatrix} i_i \\ j_i \\ k_i \end{Bmatrix}$$

$$\begin{Bmatrix} i_F \\ j_F \\ k_F \end{Bmatrix} =
\underbrace{\begin{bmatrix} \cos\Phi_F & 0 & \sin\Phi_F \\ 0 & 1 & 0 \\ -\sin\Phi_F & 0 & \cos\Phi_F \end{bmatrix}}_{[13]}
\underbrace{\begin{bmatrix} \cos\Psi_F & \sin\Psi_F & 0 \\ -\sin\Psi_F & \cos\Psi_F & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[12]}
\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda & \sin\lambda \\ 0 & -\sin\lambda & \cos\lambda \end{bmatrix}}_{[11]}
\underbrace{\begin{bmatrix} \cos\Phi_i & 0 & \sin\Phi_i \\ 0 & 1 & 0 \\ -\sin\Phi_i & 0 & \cos\Phi_i \end{bmatrix}}_{[19]}$$

THE ABOVE X_F, Y_F, Z_F ARE THE NEW EARTH-VEHICLE GEOCENTRIC AXES

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THE ORIENTATION OF THE X_F, Y_F, Z_F AXES RELATIVE TO THE EARTH ROTATING AXES X_E, Y_E, Z_E, IS OBTAINED AS FOLLOWS: FROM FIGURE 2 WE HAVE THE MATRIX FORM

$$\begin{Bmatrix} i_E \\ j_E \\ k_E \end{Bmatrix} = \begin{bmatrix} \cos\Omega_E t & \sin\Omega_E t & 0 \\ -\sin\Omega_E t & \cos\Omega_E t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} i \\ j \\ k \end{Bmatrix}$$

[Ⓘ]

TRANSPOSITION OF THIS MATRIX, AS SHOWN IN FIGURE 7, RESULTS IN THE MATRIX FORM

$$\begin{Bmatrix} i \\ j \\ k \end{Bmatrix} = \begin{bmatrix} \cos\Omega_E t & -\sin\Omega_E t & 0 \\ \sin\Omega_E t & \cos\Omega_E t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} i_E \\ j_E \\ k_E \end{Bmatrix}$$

[Ⓘ]'
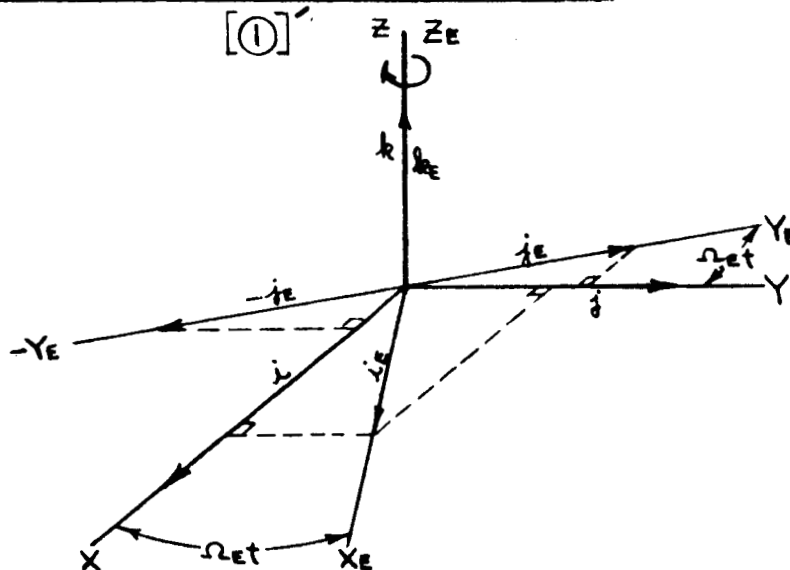
FIGURE 7

$$i = i_E \cos\Omega_E t - j_E \sin\Omega_E t + k_E(0)$$

$$j = i_E \sin\Omega_E t + j_E \cos\Omega_E t + k_E(0)$$

$$k = i_E(0) + j_E(0) + k_E(1)$$

NOW PERFORMING MATRIX MULTIPLICATION OF STEP NO. 5 MATRIX WITH THE TRANSFORMATION MATRIX OF FIGURE 3 OBTAINED ABOVE WE OBTAIN THE FINAL RESULTANT MATRIX:

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$$
\begin{Bmatrix} \lambda_E \\ \dot{\phi}_E \\ \dot{R}_E \end{Bmatrix} =
$$

$$
\begin{bmatrix}
\cos\Omega_E t & -\sin\Omega_E t & 0 \\
\sin\Omega_E t & \cos\Omega_E t & 0 \\
0 & 0 & 1
\end{bmatrix}_{(1)}
\begin{bmatrix}
\cos\phi_i & 0 & \sin\phi_i \\
0 & 1 & 0 \\
-\sin\phi_i & 0 & \cos\phi_i
\end{bmatrix}_{(10)}
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos\lambda & \sin\lambda \\
0 & -\sin\lambda & \cos\lambda
\end{bmatrix}_{(11)}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}_{(12)}
\begin{bmatrix}
\cos\phi_F & 0 & \sin\phi_F \\
0 & 1 & 0 \\
-\sin\phi_F & 0 & \cos\phi_F
\end{bmatrix}_{(13)}
\begin{Bmatrix} \dot{\lambda}_F \\ \dot{\phi}_F \\ \dot{R}_F \end{Bmatrix}
$$
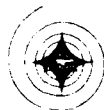
THE ORIENTATION OF THE VEHICLE BODY AXES, X_B, Y_B, Z_B, RELATIVE TO THE NEW GEOCENTRIC AXES IS DETERMINED BY MEANS OF EULER ANGLES ψ, ϕ, θ SIMILAR TO THOSE USED PREVIOUSLY, EXCEPT THAT THEY ARE NOW REFERRED TO THE NEW GEOCENTRIC AXES. THUS, SINCE X_F, Y_F, Z_F AXES TO X_V, Y_V, Z_V AXES HAS THE SAME RESULT AS X_G, Y_G, Z_G AXES TO X_V, Y_V, Z_V AXES, FROM FIGURE 4 PAGE 6 AND FIGURE 5 PAGE 7 WE OBTAIN:

$$
\begin{Bmatrix} i_B \\ j_B \\ k_B \end{Bmatrix} =
\underbrace{\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}}_{[7]}
\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}}_{[6]}
\underbrace{\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[5]}
\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{[4]}
\begin{Bmatrix} i_F \\ j_F \\ k_F \end{Bmatrix}
$$

IN THE COURSE OF THE ANALYSIS IT WILL BE DESIRABLE TO DETERMINE THE GEOCENTRIC LATITUDE Φ AND LONGITUDE Ψ. A RELATIONSHIP BETWEEN Φ AND Ψ ON THE ONE HAND AND Φ_F AND Ψ_F ON THE OTHER IS THUS REQUIRED. WE PROCEED TO ESTABLISH SUCH A RELATIONSHIP BY WRITING THE TRANSFORMATION,

$$
\begin{Bmatrix} i_F \\ j_F \\ k_F \end{Bmatrix} = [13][12][11][10][1]'[2]'[3]' \begin{Bmatrix} i_G \\ j_G \\ k_G \end{Bmatrix}
$$

SINCE $i_F = i_G$, WE MAY WRITE

$$
\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = [13][12][11][10][1]'[2]'[3]' \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}
$$

OR

$$
[1]'[2]'[3]' \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = [10]'[11]'[12]'[13]' \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}
$$

WHICH YIELDS THE FOLLOWING RELATIONS:

(1) $\cos(\Psi + \Omega_E t)\cos\Phi = \cos\Phi_i \cos\Psi_F \cos\Phi_F$

$\qquad - \sin\Phi_i \sin\lambda \sin\Psi_F \cos\Phi_F - \sin\Phi_i \cos\lambda \sin\Phi_F$

(2) $\sin(\Psi + \Omega_E t)\cos\Phi = \cos\lambda \sin\Psi_F \cos\Phi_F - \sin\lambda \sin\Phi_F$

(3) $\sin\Phi = \sin\Phi_i \cos\Psi_F \cos\Phi_F + \cos\Phi_i \sin\lambda \sin\Psi_F \cos\Phi_F$

$\qquad + \cos\Phi_i \cos\lambda \sin\Phi_F$

FROM WHICH THE COORDINATES Ψ AND Φ MAY BE DETERMINED.

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MODEL ONE

MODEL ONE IS THAT MODEL WHICH SHOWS THE RELATIONSHIP OF ROTATION BETWEEN THE FOLLOWING AXES: INERTIAL (X, Y, Z), EARTH (X_E, Y_E, Z_E), EARTH-VEHICLE GEOCENTRIC (X_G, Y_G, Z_G), VEHICLE BODY (X_B, Y_B, Z_B), VEHICLE GEOCENTRIC (X_V, Y_V, Z_V), AND VEHICLE-WIND (X_W, Y_W, Z_W).

NOTE: MODEL ONE DOES NOT INCLUDE A POLAR ORBIT.

DIFFERENTIAL EQUATIONS FOR EULER ANGLES OF MODEL ONE

A SET OF DIFFERENTIAL EQUATIONS GOVERNING THE EULER ANGLES, θ, ϕ, ψ, IS NOW FORMULATED.

THE ANGULAR VELOCITY, $\bar{\omega}_B$, OF THE VEHICLE RELATIVE TO THE INERTIAL AXES MAY BE WRITTEN AS FOLLOWS:

$$\bar{\omega}_B = P\, \imath_B + Q\, \jmath_B + R\, k_B \tag{1}$$

WHERE P, Q AND R ARE THE COMPONENTS ABOUT THE X_B, Y_B, Z_B AXES RESPECTIVELY. THIS EQUATION MAY BE WRITTEN IN THE MATRIX FORM:

$$\left\{\bar{\omega}_B\right\}_B = \begin{Bmatrix} P \\ Q \\ R \end{Bmatrix} \tag{2}$$

IN WHICH THE ELEMENTS ARE COMPONENTS OF THE VECTOR AND THE SUBSCRIPT OUTSIDE THE BRACKET IDENTIFIES THE AXES SYSTEM WITH RESPECT TO WHICH THESE COMPONENTS ARE TAKEN. WE NOW PROCEED TO RELATE THESE COMPONENTS TO THE EULER ANGLES, θ, ϕ, ψ, THE GEOCENTRIC COORDINATES Φ AND Ψ, AND THEIR DERIVATIVES. IN DOING THIS WE SET UP AN ALTERNATIVE REPRESENTATIVE FOR $\bar{\omega}_B$.

WE FIRST WRITE, IN MATRIX FORM, THE ANGULAR VELOCITY, $\bar{\omega}_E$, OF THE EARTH ROTATING AXES, X_E, Y_E, Z_E, RELATIVE TO THE INERTIAL AXES, BUT RESOLVED ABOUT THE EARTH ROTATING AXES. THUS:

$$\left\{\bar{\omega}_E\right\} = [1] \begin{Bmatrix} 0 \\ 0 \\ \Omega_E \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \Omega_E \end{Bmatrix} \tag{3}$$

THE ANGULAR VELOCITY OF THE EARTH-VEHICLE GEOCENTRIC AXES, X_G, Y_G, Z_G, RELATIVE TO THE INERTIAL AXES, BUT RESOLVED ABOUT EARTH-VEHICLE GEOCENTRIC AXES IS:

$$\left\{\bar{\omega}_G\right\}_G = [3][2]\left\{\bar{\omega}_E\right\}_E + [3][2] \begin{Bmatrix} 0 \\ 0 \\ \dot{\Psi} \end{Bmatrix} + [3] \begin{Bmatrix} 0 \\ -\dot{\Phi} \\ 0 \end{Bmatrix}$$

OR: $\left\{\varpi_G\right\}_G = [3] \left\{\begin{matrix} 0 \\ 0 \\ \Omega_E + \dot{\psi} \end{matrix}\right\} + \left\{\begin{matrix} 0 \\ -\dot{\phi} \\ 0 \end{matrix}\right\}$ (4)

THE ANGULAR VELOCITY OF THE VEHICLE GEOCENTRIC AXES, X_V, Y_V, Z_V, RELATIVE TO THE INERTIAL AXES, BUT RESOLVED ABOUT THE VEHICLE GEOCENTRIC AXES IS:

$\left\{\varpi_V\right\}_V = [4]\left\{\varpi_G\right\}_G = [4][3]\left\{\begin{matrix} 0 \\ 0 \\ \Omega_E + \dot{\psi} \end{matrix}\right\} + [4]\left\{\begin{matrix} 0 \\ -\dot{\phi} \\ 0 \end{matrix}\right\}$ (5)

FINALLY, THE ANGULAR VELOCITY OF THE VEHICLE BODY AXES, X_B, Y_B, Z_B, RELATIVE TO THE INERTIAL AXES, BUT RESOLVED ABOUT THE VEHICLE BODY AXES IS:

$\left\{\varpi_B\right\}_B = [7][6][5]\left\{\varpi_V\right\}_V + [7][6][5]\left\{\begin{matrix} 0 \\ 0 \\ \dot{\psi} \end{matrix}\right\} + [7][6]\left\{\begin{matrix} \dot{\phi} \\ 0 \\ 0 \end{matrix}\right\} + [7]\left\{\begin{matrix} 0 \\ \dot{\theta} \\ 0 \end{matrix}\right\}$

OR:

$\left\{\varpi_B\right\}_B = [7][6][5][4][3]\left\{\begin{matrix} 0 \\ 0 \\ \Omega_E + \dot{\psi} \end{matrix}\right\} + [7][6][5][4]\left\{\begin{matrix} 0 \\ -\dot{\phi} \\ 0 \end{matrix}\right\}$

$\qquad + [7][6]\left\{\begin{matrix} 0 \\ 0 \\ \dot{\psi} \end{matrix}\right\} + [7]\left\{\begin{matrix} \dot{\phi} \\ 0 \\ 0 \end{matrix}\right\} + \left\{\begin{matrix} 0 \\ \dot{\theta} \\ 0 \end{matrix}\right\}$ (6)

BY EQUATING AND REDUCING EQUATIONS (2) AND (6) WE HAVE:

$\left\{\begin{matrix} P \\ Q \\ R \end{matrix}\right\} = [7][6][5][4][3]\left\{\begin{matrix} 0 \\ 0 \\ \Omega_E + \dot{\psi} \end{matrix}\right\} + [7][6][5][4]\left\{\begin{matrix} 0 \\ -\dot{\phi} \\ 0 \end{matrix}\right\}$

$\qquad + \begin{bmatrix} \cos\theta & 0 & -\sin\theta\cos\phi \\ 0 & 1 & \sin\phi \\ \sin\theta & 0 & \cos\theta\cos\phi \end{bmatrix} \left\{\begin{matrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{matrix}\right\}$ (7)

OR:

$\left\{\begin{matrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{matrix}\right\} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta\cos\phi \\ 0 & 1 & \sin\phi \\ \sin\theta & 0 & \cos\theta\cos\phi \end{bmatrix}^{-1} \left\{ \left\{\begin{matrix} P \\ Q \\ R \end{matrix}\right\} - [7][6][5][4][3]\left\{\begin{matrix} 0 \\ 0 \\ \Omega_E + \dot{\psi} \end{matrix}\right\} \right.$

$\qquad \left. - [7][6][5][4]\left\{\begin{matrix} 0 \\ -\dot{\phi} \\ 0 \end{matrix}\right\} \right\}$ (8)

PERFORMING THE INDICATED MATRIX INVERSION AND MATRIX MULTIPLICATIONS, AND REDUCING, WE FINALLY OBTAIN IN SCALAR FORM THE EQUATIONS:

$$\dot{\theta} = P \sin\theta \tan\phi + Q - R\cos\theta\tan\phi + \dot{\phi}\cos\psi \sec\phi + (\Omega_E + \dot{\Psi})\cos\Phi \sin\psi \sec\phi \tag{9}$$

$$\dot{\phi} = P\cos\theta + R\sin\theta + \dot{\phi}\sin\psi - (\Omega_E + \dot{\Psi})\cos\Phi\cos\psi \tag{10}$$

$$\dot{\psi} = -P\sin\theta\sec\phi + R\cos\theta\sec\phi - \dot{\phi}\cos\psi\tan\phi$$
$$- (\Omega_E + \dot{\Psi})\cos\Phi\sin\psi\tan\phi + (\Omega_E + \dot{\Psi})\sin\Phi \tag{11}$$

TRANSLATIONAL EQUATIONS OF MOTION

LET THE RADIUS VECTOR FROM THE EARTH'S CENTER TO THE VEHICLE CENTROID BE $\bar{r}$. THEN THE VELOCITY VECTOR RELATIVE TO THE INERTIAL AXES IS GIVEN BY:

$$\nabla = \frac{d\bar{r}}{dt} = \left(\frac{\delta\bar{r}}{\delta t}\right)_G + \bar{\omega}_G \times \bar{r} \tag{12}$$

WHERE $\left(\frac{\delta}{\delta t}\right)_G$ DENOTES A PARTIAL DIFFERENTIATION IN WHICH i_G, j_G, k_G, ARE HELD FIXED,

$$\text{WITH } \bar{r} = r i_G \quad \text{WHERE } r = |\bar{r}| \tag{13}$$

FROM EQUATION (4):

$$\bar{\omega}_G = (\Omega_E + \dot{\Psi})\sin\Phi\, i_G - \dot{\Phi} j_G + (\Omega_E + \dot{\Psi})\cos\Phi\, k_G \tag{14}$$

EQUATION (12) BECOMES:

$$\nabla = \dot{r} i_G + r(\Omega_E + \dot{\Psi})\cos\Phi\, j_G + r\dot{\Phi} k_G \tag{15}$$

THE ACCELERATION VECTOR RELATIVE TO THE INERTIAL AXES MAY NOW BE WRITTEN AS FOLLOWS:

$$\bar{a} = \frac{d\nabla}{dt} = \left(\frac{\delta\nabla}{\delta t}\right)_G + \bar{\omega}_G \times \nabla$$

$$\bar{a} = \left[\ddot{r} - r\left\{\dot{\Phi}^2 + (\Omega_E + \dot{\Psi})^2\cos^2\Phi\right\}\right] i_G$$

$$+ \left[\left\{2\dot{r}(\Omega_E + \dot{\Psi}) + r\ddot{\Psi}\right\}\cos\Phi - 2r\dot{\Phi}(\Omega_E + \dot{\Psi})\sin\Phi\right] j_G \tag{16}$$

$$+ \left[r\ddot{\Phi} + 2\dot{r}\dot{\Phi} + r(\Omega_E + \dot{\Psi})^2\sin\Phi\cos\Phi\right] k_G$$

THE GRAVITY ACCELERATION VECTOR, INCLUDING THE EFFECT OF EARTH OBLATENESS IS WRITTEN AS FOLLOWS:

$$\frac{\overline{G}}{m} = \left[-\frac{K}{r^2} + \frac{6\mu K R_0^2}{r^4}(2 - 3\cos^2\phi)\right]i_G + \left[-\frac{12\mu K R_0^2}{r^4}\sin\phi\cos\phi\right]k_G \quad (17)$$

WHERE G IS THE GRAVITY FORCE VECTOR, m IS THE VEHICLE MASS, R_0 IS THE RADIUS OF THE EARTH AT THE EQUATOR ($R_0 = 20,926,428$ FEET), AND K AND μ ARE GRAVITY CONSTANTS WITH THE FOLLOWING VALUES:

$$K = 0.1407750 \times 10^{17} \, FT^3 \, SEC^{-2}$$
$$6\mu = 1.638 \times 10^{-3}$$

THE AERODYNAMIC, PROPULSIVE AND CONTROL FORCES ARE FIRST COMPUTED WITH REFERENCE TO VEHICLE BODY AXES, AND A TRANSFORMATION TO EARTH-VEHICLE GEOCENTRIC AXES IS THEN EFFECT.

THE AERODYNAMIC FORCE VECTOR IS WRITTEN AS FOLLOWS:

$$\overline{F} = F_x i_B + F_y j_B + F_z k_B$$
$$= F_r i_G + F_\psi j_G + F_\phi k_G \quad (18)$$

THE PROPULSIVE FORCE VECTOR IS:

$$\overline{P} = P_x i_B + P_y j_B + P_z k_B$$
$$= P_r i_G + P_\psi j_G + P_\phi k_G \quad (19)$$

AND THE CONTROL FORCE VECTOR IS:

$$\overline{H} = H_x i_B + H_y j_B + H_z k_B$$
$$= H_r i_G + H_\psi j_G + H_\phi k_B \quad (20)$$

SUMMING CORRESPONDING COMPONENTS FROM EQUATIONS (18), (19) AND (20), THE FOLLOWING MATRIX EQUATION REPRESENTS THE TRANSFORMATION FROM BODY AXES TO EARTH GEOCENTRIC AXES

$$\begin{Bmatrix} F_r + P_r + H_r \\ F_\psi + P_\psi + H_\psi \\ F_\phi + P_\phi + H_\phi \end{Bmatrix} = [4]'[5]'[6]'[7]' \begin{Bmatrix} F_x + P_x + H_x \\ F_y + P_y + H_y \\ F_z + P_z + H_z \end{Bmatrix} \quad (21)$$

WHERE THE PRIME DENOTES MATRIX TRANSPOSITION. UPON MULTIPLICATION, THE TRANSFORMATION MATRIX BECOMES:

$$[\textcircled{4}\,\textcircled{5}\,\textcircled{6}\,\textcircled{7}] =$$

$$
\begin{bmatrix}
\sin\theta\cos\phi & -\sin\phi & -\cos\theta\cos\phi \\
\sin\psi\cos\theta+\cos\psi\sin\theta\sin\phi & \cos\psi\cos\phi & \sin\psi\sin\theta-\cos\psi\cos\theta\sin\phi \\
\cos\psi\cos\theta-\sin\psi\sin\theta\sin\phi & -\sin\psi\cos\phi & \cos\psi\sin\theta+\sin\psi\cos\theta\sin\phi
\end{bmatrix}
$$

(22)

THE CORRESPONDING TRANSFORMATION MATRIX IN THE CASE OF CONVENTIONAL EULER ANGLES IS OBTAINED BY INTERCHANGING MATRICES $[\textcircled{6}]'$ AND $[\textcircled{7}]'$ IN EQUATION (22).

SUMMING APPLIED, GRAVITY AND INERTIA FORCE COMPONENTS IN THE DIRECTION OF VEHICLE-GEOCENTRIC AXES, WE HAVE THE FOLLOWING THREE TRANSLATIONAL EQUATIONS OF MOTION IN TERMS OF THE COORDINATES r, Ψ, AND Φ:

$$\ddot{r}-r\left\{\dot{\Phi}^2+(\Omega_E+\dot{\Psi})^2\cos^2\Phi\right\}=-\frac{K}{r^2}+\frac{6\nu KR_0^2}{r^4}(2-3\cos^2\Phi)+\frac{1}{m}\left(F_r+P_r+H_r\right) \quad (23)$$

$$\left\{r\ddot{\Psi}+2\dot{r}(\Omega_E+\dot{\Psi})\right\}\cos\Phi-2r\dot{\Phi}(\Omega_E+\dot{\Psi})\sin\Phi=\frac{1}{m}\left(F_\Psi+P_\Psi+H_\Psi\right) \quad (24)$$

$$r\ddot{\Phi}+2\dot{r}\dot{\Phi}+r(\Omega_E+\dot{\Psi})^2\sin\Phi\cos\Phi=-\frac{12\nu KR_0^2}{r^4}\sin\Phi\cos\Phi+\frac{1}{m}\left(F_\Phi+P_\Phi+H_\Phi\right) \quad (25)$$

GIVEN THE SHAPE OF THE OBLATE EARTH, APPROXIMATELY AS FOLLOWS:

$$R_E=R_0(1-f\sin^2\Phi) \quad (26)$$

WHERE R_E IS THE DISTANCE FROM THE EARTH'S CENTER TO A LOCAL POINT ON THE EARTH'S SURFACE, AND

$$f=0.0033670034$$

THE ALTITUDE h CAN BE DETERMINED FROM THE RELATION,

$$h=r-R_0(1-f\sin^2\Phi) \quad (27)$$

IT CAN BE SEEN FROM EQUATION (24) THAT Ψ AND ITS DERIVATIVES ARE INDETERMINATE AT $\Phi=90°$, THUS PRECLUDING THE USE OF THE PRESENT EQUATIONS FOR SIMULATION OF FLIGHT OVER A POLE.

ROTATIONAL EQUATIONS OF MOTION

THE ROTATIONAL EQUATIONS OF MOTION DEVELOPED ON THE BASIS OF MOMENT EQUILIBRIUM ABOUT THE BODY AXES ARE THE SAME AS THOSE FAMILIAR IN AIRCRAFT ANALYSIS. FOR A VEHICLE WITH THE X_B-Z_B PLANE A PLANE OF SYMMETRY, THEY ARE:

$$-\left[\dot{P}I_{xx}-(I_{yy}-I_{zz})QR-I_{xz}(\dot{R}+PQ)\right]+L+T_x+J_x=0 \tag{28}$$

$$-\left[\dot{Q}I_{yy}-(I_{zz}-I_{xx})RP-I_{xz}(R^2-P^2)\right]+M+T_y+J_y=0 \tag{29}$$

$$-\left[\dot{R}I_{zz}-(I_{xx}-I_{yy})PQ-I_{xz}(\dot{P}-QR)\right]+N+T_z+J_z=0 \tag{30}$$

WHERE $I_{xx}, I_{yy}, I_{zz}, I_{xz}$ ARE MOMENTS AND PRODUCTS OF INERTIA REFERRED TO THE BODY AXES, L, M, AND N ARE COMPONENTS OF THE AERODYNAMIC MOMENT, T_x, T_y, AND T_z ARE COMPONENTS OF THE PROPULSIVE MOMENT, AND J_x, J_y AND J_z ARE COMPONENTS OF THE CONTROL MOMENT, ALL REFERRED TO THE X_B, Y_B AND Z_B AXES RESPECTIVELY.

ANGLE OF ATTACK AND ANGLE OF SIDESLIP

WITH APPROPRIATE MODIFICATION OF EQUATION (15), THE VELOCITY OF THE VEHICLE RELATIVE TO THE EARTH ROTATING AXES, X_E, Y_E, Z_E, BECOMES:

$$\bar{V}_E = \dot{r}\,i_G + r\dot{\Psi}\cos\Phi\,j_G + r\dot{\Phi}\,k \tag{31}$$

IF WE NOW ALLOW A WIND VELOCITY GIVEN BY

$$\bar{V}_w = V_{w_r}i_G + V_{w_\Psi}j_G + V_{w_\Phi}k_G \tag{32}$$

THE VELOCITY OF THE VEHICLE RELATIVE TO THE AIR BECOMES

$$\bar{V}_a = V_{a_r}i_G + V_{a_\Psi}j_G + V_{a_\Phi}k_G \tag{33}$$

WHERE $V_{a_r} = \dot{r} - V_{w_r}$

$$V_{a_\Psi} = r\dot{\Psi}\cos\Phi - V_{w_\Psi}$$

$$V_{a_\Phi} = r\dot{\Phi} - V_{w_\Phi}$$

AND THE MAGNITUDE OF THIS VELOCITY IS:

$$V_a = \sqrt{V_{a_r}^2 + V_{a_\Psi}^2 + V_{a_\Phi}^2} \tag{34}$$

A TRANSFORMATION TO BODY AXES MAY BE EFFECTED AS FOLLOWS:

$$\begin{Bmatrix} V_{a_x} \\ V_{a_y} \\ V_{a_z} \end{Bmatrix} = \boxed{7}\boxed{6}\boxed{5}\boxed{4} \begin{Bmatrix} V_{a_r} \\ V_{a_\Psi} \\ V_{a_\Phi} \end{Bmatrix} \tag{35}$$

NOTING THAT

$$\overline{V_a} = V_a \, \iota_w \tag{36}$$

A TRANSFORMATION FROM WIND AXES TO BODY AXIS IS GIVEN BY:

$$\begin{Bmatrix} V_{a_x} \\ V_{a_y} \\ V_{a_z} \end{Bmatrix} = \boxed{8}\boxed{9} \begin{Bmatrix} V_a \\ 0 \\ 0 \end{Bmatrix} = V_a \begin{Bmatrix} \cos\alpha \cos\beta \\ \sin\beta \\ \sin\alpha \cos\beta \end{Bmatrix} \tag{37}$$

EQUATING THE RIGHT HAND SIDES OF EQUATIONS (35) AND (37), WE HAVE:

$$\begin{Bmatrix} \cos\alpha \cos\beta \\ \sin\beta \\ \sin\alpha \cos\beta \end{Bmatrix} = \boxed{7}\boxed{6}\boxed{5}\boxed{4} \begin{Bmatrix} V_{a_r}/V_a \\ V_{a_\Psi}/V_a \\ V_{a_\Phi}/V_a \end{Bmatrix} \tag{38}$$

FROM WHICH α AND β MAY BE DETERMINED. THE TRANSFORMATION MATRIX IN THIS EQUATION IS SEEN TO BE THE TRANSPOSE OF THE MATRIX GIVEN IN EQUATION (22). THE CORRESPONDING TRANSFORMATION MATRIX IN THE CASE OF CONVENTIONAL EULER ANGLES IS OBTAINED BY INTERCHANGING MATRICES $\boxed{6}$ AND $\boxed{7}$ IN EQUATION (38).

MODEL TWO

MODEL TWO IS THAT MODEL WHICH SHOWS THE RELATIONSHIP OF ROTATION BETWEEN THE FOLLOWING AXES:

INERTIAL (X, Y, Z), EARTH (X_E, Y_E, Z_E), ORIGINAL EARTH-VEHICLE GEOCENTRIC (X_G, Y_G, Z_G), NEW EARTH-VEHICLE GEOCENTRIC (X_F, Y_F, Z_F), VEHICLE BODY (X_B, Y_B, Z_B), VEHICLE GEOCENTRIC (X_V, Y_V, Z_V) AND WIND-VEHICLE (X_W, Y_W, Z_W).

NOTE: MODEL TWO INCLUDES A POLAR ORBIT.

DIFFERENTIAL EQUATIONS FOR THE EULER ANGLES

FOLLOWING THE NOTATIONS AND EQUATIONS (1) THRU (11) ON PAGE 23, 24 AND 25, THE ANGULAR VELOCITY OF THE X_F, Y_F, Z_F, AXES RELATIVE TO THE INERTIAL AXES, BUT RESOLVED ABOUT THE X_F, Y_F, Z_F AXES, IS GIVEN BY:

$$\{\overline{\omega}_F\}_F = [13] \begin{Bmatrix} 0 \\ 0 \\ \dot{\Psi}_F \end{Bmatrix} + \begin{Bmatrix} 0 \\ -\dot{\Phi}_F \\ 0 \end{Bmatrix} \tag{39}$$

SIMILARLY, THE ANGULAR VELOCITY OF THE BODY AXES RELATIVE TO THE INERTIAL AXES, BUT RESOLVED ABOUT THE BODY AXES IS GIVEN BY:

$$\{\overline{\omega}_B\}_B = [7][6][5][4][13] \begin{Bmatrix} 0 \\ 0 \\ \dot{\Psi}_F \end{Bmatrix} + [7][6][5][4] \begin{Bmatrix} 0 \\ -\dot{\Phi}_F \\ 0 \end{Bmatrix}$$

$$+ [7][6] \begin{Bmatrix} 0 \\ 0 \\ \dot{\psi} \end{Bmatrix} + [7] \begin{Bmatrix} \dot{\phi} \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \dot{\theta} \\ 0 \end{Bmatrix} \tag{40}$$

BUT,

$$\{\overline{\omega}_B\}_B = \begin{Bmatrix} P \\ Q \\ R \end{Bmatrix} \tag{41}$$

THEREFORE, EQUATING THE RIGHT-HAND SIDES OF EQUATIONS (40) AND (41) AND PROCEEDING AS ON PAGE 23, 24 AND 25, WE OBTAIN THE RELATIONS:

$$\dot{\theta} = P \sin\theta \tan\phi + Q - R \cos\theta \tan\phi + \dot{\Phi}_F \cos\psi \sec\phi + \dot{\Psi}_F \cos\Phi_F \sin\psi \sec\phi \tag{42}$$

$$\dot{\phi} = P \cos\theta + R \sin\theta + \dot{\Phi}_F \sin\psi - \dot{\Psi}_F \cos\Phi_F \cos\psi \tag{43}$$

$$\dot{\psi} = P \sin\theta \sec\phi + R \cos\theta \sec\phi - \dot{\Phi}_F \cos\psi \tan\phi - \dot{\Psi}_F \cos\Phi_F \sin\psi \tan\phi + \dot{\Psi}_F \sin\Phi_F \tag{44}$$

TRANSLATIONAL EQUATIONS OF MOTION

FOLLOWING THE ANALYSIS ON PAGES 25 THRU 27, THE RADIUS VECTOR, VELOCITY AND ACCELERATION MAY BE WRITTEN AS FOLLOWS:

$$\bar{r} = r\, i_F \tag{45}$$

$$\nabla = \dot{r}\, i_F + r\,\dot{\Psi}_F \cos\Phi_F\, j_F + r\,\dot{\Phi}_F\, k_F \tag{46}$$

$$\bar{a} = \left[\ddot{r} - r(\dot{\Phi}_F^2 + \dot{\Psi}_F^2 \cos^2\Phi_F)\right] i_F + \left[(2\dot{r}\,\dot{\Psi}_F + r\,\ddot{\Psi}_F)\cos\Phi_F - 2r\,\dot{\Phi}_F\,\dot{\Psi}_F \sin\Phi_F\right] j_F$$

$$+ \left[r\,\ddot{\Phi}_F + 2\dot{r}\,\dot{\Phi}_F + r\,\dot{\Psi}_F^2 \sin\Phi_F \cos\Phi_F\right] k_F \tag{47}$$

THE GRAVITY ACCELERATION VECTOR MUST NOW BE RESOLVED INTO COMPONENTS ALONG THE X_F, Y_F, Z_F AXES. THIS NECESSITATES A TRANSFORMATION FROM THE ORIGINAL TO THE NEW EARTH-VEHICLE GEOCENTRIC AXES. SINCE THE X_F-AXIS IS COINCIDENT WITH THE X_G-AXIS, THIS TRANSFORMATION INVOLVE SIMPLY A ROTATION ABOUT THE X_G-AXIS THROUGH AN ANGLE WHICH WE WILL DENOTE BY λ_F. THUS:

$$\begin{Bmatrix} i_F \\ j_F \\ k_F \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda_F & \sin\lambda_F \\ 0 & -\sin\lambda_F & \cos\lambda_F \end{bmatrix} \begin{Bmatrix} i_G \\ j_G \\ k_G \end{Bmatrix} \tag{48}$$

AND

$$\frac{\bar{G}}{m} = \left[-\frac{K}{r^2} + \frac{6\mu K R_0^2}{r^4}(2-3\cos^2\Phi) \right] i_F$$

$$+ \left[-\frac{12K\mu R_0^2}{r^4} \sin\lambda_F \sin\Phi \cos\Phi \right] j_F$$

$$+ \left[-\frac{12K\mu R_0^2}{r^4} \cos\lambda_F \sin\Phi \cos\Phi \right] k_F \tag{49}$$

FROM EQUATION ON PAGE 21 AND EQUATION (48) WE CAN WRITE

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda_F & \sin\lambda_F \\ 0 & -\sin\lambda_F & \cos\lambda_F \end{bmatrix} = \left[\text{⑬}\right]\left[\text{⑫}\right]\left[\text{⑪}\right]\left[\text{⑩}\right]\left[\text{①}\right]\left[\text{②}\right]\left[\text{③}\right] \tag{50}$$

OR, INVERTING,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda_F & -\sin\lambda_F \\ 0 & \sin\lambda_F & \cos\lambda_F \end{bmatrix} = \boxed{3}\,\boxed{2}\,\boxed{1}\,\boxed{10}\,\boxed{11}\,\boxed{12}\,\boxed{13} \tag{51}$$

THIS MAY BE REARRANGED IN THE FORM,

$$\boxed{2}\,\boxed{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda_F & -\sin\lambda_F \\ 0 & \sin\lambda_F & \cos\lambda_F \end{bmatrix} = \boxed{1}\,\boxed{10}\,\boxed{11}\,\boxed{12}\,\boxed{13} \tag{52}$$

EQUATING THE ELEMENTS IN THE LAST ROW AND SECOND AND THIRD COLUMNS OF THE PRODUCT MATRICES ON BOTH SIDES OF EQUATION (52), WE HAVE FINALLY,

$$\cos\Phi \sin\lambda_F = -\sin\Phi_i \sin\Psi_F + \cos\Phi_i \sin\lambda \cos\Psi_F \tag{53}$$

$$\cos\Phi \cos\lambda_F = -\sin\Phi_i \sin\Phi_F \cos\Psi_F - \cos\Phi_i \sin\lambda \sin\Phi_F \sin\Psi_F$$

$$+ \cos\Phi_i \cos\lambda \cos\Phi_F \tag{54}$$

WE CAN NOW SUBSTITUTE EQUATIONS (53) AND (54) INTO EQUATION (49) TO OBTAIN THE GRAVITY ACCELERATION IN THE FORM,

$$\frac{\overline{G}}{m} = \left[-\frac{K}{r^2} + \frac{6\mu K R_o^2}{r^4}(2-3\cos^2\Phi) \right] i_F$$

$$+ \left[\frac{12 K \mu R_o^2}{r^4} \sin\Phi \left(\sin\Phi_i \sin\Psi_F - \cos\Phi_i \sin\lambda \cos\Psi_F\right) \right] j_F$$

$$+ \left[\frac{12 K \mu R_o^2}{r^4} \sin\Phi \left(\sin\Phi_i \sin\Phi_F \cos\Psi_F + \cos\Phi_i \sin\lambda \sin\Phi_F \sin\Psi_F\right.\right.$$

$$\left.\left. - \cos\Phi_i \cos\lambda \cos\Phi_F\right) \right] k_F \tag{55}$$

THE AERODYNAMIC, PROPULSIVE AND CONTROL FORCES ARE AGAIN DETERMINED WITH REFERENCE TO BODY AXES AND THEN TRANSFORMED TO VEHICLE GEOCENTRIC AXES, X_F, Y_F, Z_F, USING THE TRANSFORMATION MATRIX OF EQUATION (21), BUT RECOGNIZING THAT THE EULER ANGLES HERE ARE REFERRED TO THE X_F, Y_F, Z_F, AXES.

THE TRANSLATIONAL EQUATIONS OF MOTION MAY NOW BE WRITTEN AS FOLLOWS:

$$\ddot{r} - r(\dot{\Phi}_F^2 + \dot{\Psi}_F^2 \cos^2 \Phi_F) = -\frac{K}{r^2} + \frac{6\mu KR_o^2}{r^4}(2 - 3\cos^2 \Phi) + \frac{1}{m}(F_r + P_r + H_r) \qquad (56)$$

$$r\ddot{\Psi}_F \cos \Phi_F + 2\dot{r}\dot{\Psi}_F \cos \Phi_F - 2r\dot{\Phi}_F \dot{\Psi}_F \sin \Phi_F$$

$$= \frac{12 K\mu R_o^2}{r^4} \sin \Phi (\sin \Phi_L \sin \Psi_F - \cos \Phi_L \sin \lambda \cos \Psi_F)$$

$$+ \frac{1}{m}(F_{\Psi_F} + P_{\Psi_F} + H_{\Psi_F}) \qquad (57)$$

$$r\ddot{\Phi}_F + 2\dot{r}\dot{\Phi}_F + r\dot{\Psi}_F^2 \sin \Phi_F \cos \Phi_F = \frac{12 K\mu R_o^2}{r^4} \sin \Phi (\sin \Phi_L \sin \Phi_F \cos \Psi_F$$

$$+ \cos \Phi_L \sin \lambda \sin \Phi_F \sin \Psi_F - \cos \Phi_L \cos \lambda \cos \Phi_F)$$

$$+ \frac{1}{m}(F_{\Phi_F} + P_{\Phi_F} + H_{\Phi_F}) \qquad (58)$$

UPON SOLUTION OF THESE EQUATIONS, THE ALTITUDE MAY AGAIN BE DETERMINED FROM EQUATION (27). BECAUSE OF THE INCREASED COMPLEXITY OF THE EQUATIONS, EVEN WITH r AS A BASIC VARIABLE, A REFORMULATION TO INTRODUCE h AS A BASIC VARIABLE IS NOT CARRIED OUT. HOWEVER, THE SUBSTITUTION INDICATED IN THE EQUATION, $r = R_o + \delta r$, MAY BE MADE.

ROTATIONAL EQUATIONS OF MOTION

THE CHANGE IN COORDINATE SYSTEM DOES NOT AFFECT THE ROTATIONAL EQUATIONS OF MOTION, AND EQUATIONS (28), (29) AND (30) REMAIN APPLICABLE.

ANGLE OF ATTACK AND ANGLE OF SIDESLIP

THE VELOCITY OF THE VEHICLE RELATIVE TO THE EARTH ROTATING AXES, X_E, Y_E, Z_E, MAY BE WRITTEN,

$$\nabla_E = \left(\frac{\delta \bar{r}}{\delta t}\right)_F + \bar{\omega}_{FE} \times \bar{r} \tag{59}$$

IN WHICH $\bar{r}$ IS GIVEN BY EQUATION (45), $\left(\frac{\delta}{\delta t}\right)_F$ DENOTES A PARTIAL DIFFERENTIATION IN WHICH $\hat{\imath}_F, \hat{\jmath}_F, \hat{k}_F$ ARE HELD FIXED, AND $\bar{\omega}_{FE}$ IS THE ROTATIONAL VELOCITY OF THE X_F, Y_F, Z_F FRAME RELATIVE TO THE X_E, Y_E, Z_E FRAME. IN TERMS OF COMPONENTS ABOUT THE X_F, Y_F, Z_F AXES, $\bar{\omega}_{FE}$ IS GIVEN BY,

$$\left\{\bar{\omega}_{FE}\right\}_F = [13]\left\{\begin{matrix}0\\0\\\dot{\Psi}_F\end{matrix}\right\} + \left\{\begin{matrix}0\\-\dot{\Phi}_F\\0\end{matrix}\right\} - [13][12][11][10]\left\{\begin{matrix}0\\0\\\Omega_E\end{matrix}\right\} \tag{60}$$

OR, AFTER PERFORMING THE INDICATED MATRIX MULTIPLICATIONS,

$$\bar{\omega}_{FE} = \left\{\dot{\Psi}_F \sin\Phi_F - \Omega_E \sin\Phi_i \cos\Psi_F \cos\Phi_F\right.$$
$$\left. - \Omega_E \cos\Phi_i \sin\lambda \sin\Psi_F \cos\Phi_F - \Omega_E \cos\Phi_i \cos\lambda \sin\Phi_F\right\}\hat{\imath}_F$$
$$+ \left\{-\dot{\Phi}_F + \Omega_E \sin\Phi_i \sin\Psi_F - \Omega_E \cos\Phi_i \sin\lambda \cos\Psi_F\right\}\hat{\jmath}_F$$
$$+ \left\{\dot{\Psi}_F \cos\Phi_F + \Omega_E \sin\Phi_i \cos\Psi_F \sin\Phi_F\right.$$
$$\left. + \Omega_E \cos\Phi_i \sin\lambda \sin\Psi_F \sin\Phi_F - \Omega_E \cos\Phi_i \cos\lambda \cos\Phi_F\right\}\hat{k}_F \tag{61}$$

IF WE NOW ALLOW A WIND VELOCITY GIVEN BY,

$$\nabla_W = V_{W_r}\hat{\imath}_F + V_{W_{\Psi_F}}\hat{\jmath}_F + V_{W_{\Phi_F}}\hat{k}_F \tag{62}$$

THE VELOCITY OF THE VEHICLE RELATIVE TO THE AIR BECOMES,

$$\nabla_a = \left(\frac{\delta \bar{r}}{\delta t}\right)_F + \bar{\omega}_{FE} \times \bar{r} - \nabla_W \tag{63}$$

INTRODUCING EQUATIONS (45), (61) AND (62) INTO EQUATION (63), WE HAVE FINALLY,

$$\nabla_a = V_{a_r}\hat{\imath}_F + V_{a_{\Psi_F}}\hat{\jmath}_F + V_{a_{\Phi_F}}\hat{k}_F \tag{64}$$

WHERE

$$V_{a_r} = \dot{r} - V_{W_r}$$

$$V_{a_{\Psi_F}} = r\left(\dot{\Psi}_F \cos\Phi_F + \Omega_E \sin\Phi_i \cos\Psi_F \sin\Phi_F\right.$$
$$\left. + \Omega_E \cos\Phi_i \sin\lambda \sin\Psi_F \sin\Phi_F - \Omega_E \cos\Phi_i \cos\lambda \cos\Phi_F\right) - V_{W_{\Psi_F}}$$

$$V_{a_{\Phi_F}} = r\left(\dot{\Phi}_F - \Omega_E \sin\Phi_i \sin\Psi_F + \Omega_E \cos\Phi_i \sin\lambda \cos\Psi_F\right) - V_{W_{\Phi_F}}$$

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AND

$$V_a = \sqrt{V_{a_r}^2 + V_{a_{\Psi_F}}^2 + V_{a_{\xi_F}}^2}$$

(65)

α AND β MAY NOW BE DETERMINED AS PREVIOUSLY, ON PAGE 29, WITH EQUATION (38) BEING REPLACED BY THE RELATION

$$\begin{Bmatrix} \cos\alpha\ \cos\beta \\ \sin\beta \\ \sin\alpha\ \cos\beta \end{Bmatrix} = \boxed{7}\ \boxed{6}\ \boxed{5}\ \boxed{4} \begin{Bmatrix} V_{a_r}/V_a \\ V_{a_{\Psi_F}}/V_a \\ V_{a_{\xi_F}}/V_a \end{Bmatrix}$$

(66)

WHERE AGAIN THE TRANSFORMATION MATRIX IS THE TRANSPOSE OF THE MATRIX IN EQUATION (22).

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MATHEMATICAL MODEL FOR LAUNCH ESCAPE SYSTEM

LAUNCH ESCAPE PROPULSION SYSTEM

DEFINITIONS:

C.M. - COMMAND MODULE

L.E.S.- LAUNCH ESCAPE SYSTEM

$X_B, Y_B, Z_B,$ - COORDINATE BODY AXES OF CONFIGURATION

P - L.E.S. PROPULSION

$P_X, P_Y, P_Z,$ - L.E.S. PROPULSION ALONG COORDINATE AXES

P_{SFM} - SOLID FUEL MOTOR PROPULSION

R_{XSFM} - RADIUS ALONG X-AXIS FROM SOLID FUEL MOTOR
            TO CENTER OF GRAVITY LINE

T_X, T_Y, T_Z - TORQUE ABOUT COORDINATE AXIS

$f(FB)$ - FUNCTION OF TOTAL FUEL BURNED

C.G. - CENTER OF GRAVITY

C.G'. - CENTER OF GRAVITY AFTER FUEL BURN OUT

INTRODUCTION:

IN THE EVENT OF ABORT CONDITIONS OF THE APOLLO
MISSION DURING THE PRE-ORBIT PHASE, THE LAUNCH
ESCAPE SYSTEM IS UTILIZED TO LIFT THE COMMAND
MODULE FROM THE BOOSTER TO SAFETY.

TRAJECTORY:

IN GENERAL, THE TRAJECTORY RESULTING FROM THE
SYSTEM UTILIZATION IS PARABOLIC WITH AN ALTITUDE
OF APPROXIMATELY 6900 FT. AND A SPAN OF APPROXIMATELY
2600 FT.

PROPULSION:

AS SHOWN IN FIGURE 1, THE PRIMARY PROPULSION OF
THE SYSTEM IS PRODUCED BY A SINGLE ENGINE THROUGH
FOUR NOZZLES PLACED AT AN ANGLE OF 33° TO THE
FORWARD THRUST VECTOR. SINCE THE EXTENSION OF
THIS VECTOR PASSES THROUGH THE C.G. AT ALL TIMES, IT
MAKES AN ANGLE WITH THE X_B AXIS OF 3°15'. AS THE
FUEL BURNS AND THE C.G. SHIFTS TO ITS FINAL POSITION C.G'.
THE VECTOR EXTENDED CONTINUES ESSENTIALLY TO
PASS THROUGH THE C.G. AT ANY TIME T.

LAUNCH ESCAPE PROPULSION SYSTEM

FIGURE 1

(2)

THE INITIAL C.G. IS 6.78' FROM IMPULSE POINT OR CENTER OF THE COORDINATE AXES. AS THE FUEL BURNS, THE C.G. SHIFTS TO A POINT 10.70' AWAY, MEASURED ALONG THE X_B-AXIS. A SOLID FUEL MOTOR IS LOCATED IN THE UPPER REGION OF THE ESCAPE TOWER NORMAL TO THE X_B-AXIS. FOR FURTHER SIMPLIFICATION, THE PROPULSION VECTOR OF THIS MOTOR IS CONSIDERED DIRECTED PARALLEL TO THE Z_B-AXIS.

CONSEQUENTLY THE PROPULSIVE FORCES ALONG THE THREE AXIS CAN BE WRITTEN:

$$P_{X_{LES}} = P \cos 3° 15'$$

$$P_{Y_{LES}} = 0$$

$$P_{Z_{LES}} = P \sin 3° 15' + P_{SFM}$$

TORQUE: THE TORQUE ABOUT THE AXES ARE:

$$T_{X_{LES}} = 0$$

$$T_{Y_{LES}} = \left[6.78' + 3.92' f(FB)\right] P_{LES} \sin 3° 15'$$
$$+ \left[R_{X_{LES}} + 3.92' f(FB)\right] P_{SFM}$$

$$T_{Z_{LES}} = 0$$

PART II

MATHEMATICAL MODEL FOR MIDCOURSE ENVIRONMENT

LIST OF SYMBOLS

$\bar{a}_L$ ACCELERATION VECTOR OF VEHICLE RELATIVE TO LUNAR INERTIAL AXES.

● $\bar{a}_h$ ACCELERATION VECTOR OF VEHICLE RELATIVE TO SUN INERTIAL AXES.

g_e GRAVITATIONAL ACCELERATION AT EARTHS SURFACE.

g_L GRAVITATIONAL ACCELERATION AT MOONS SURFACE.

g_h GRAVITATIONAL ACCELERATION AT SUNS SURFACE

R_o RADIUS IN FEET OF EARTH AT EQUATOR.

R_{oL} RADIUS IN FEET OF MOON AT EQUATOR.

R_{oh} RADIUS IN FEET OF SUN AT EQUATOR.

K EARTH GRAVITATIONAL CONSTANT.

K_L MOON GRAVITATIONAL CONSTANT.

K_h SUN GRAVITATIONAL CONSTANT.

● $\bar{r}$ RADIUS VECTOR FROM EARTHS CENTER TO VEHICLE CENTROID.

$\bar{r}_L$ RADIUS VECTOR FROM MOONS CENTER TO VEHICLE CENTROID.

$\bar{r}_h$ RADIUS VECTOR FROM SUNS CENTER TO VEHICLE CENTROID.

r LENGTH OF $\bar{r}$.

r_L LENGTH OF $\bar{r}_L$.

r_h LENGTH OF $\bar{r}_h$.

Φ_F SPHERICAL ANGULAR COORDINATE OF VEHICLE MEASURED NORMAL TO NOMINAL EARTH TRAJECTORY PLANE.

Φ_{FL} SPHERICAL ANGULAR COORDINATE OF VEHICLE MEASURED NORMAL TO NOMINAL MOON TRAJECTORY PLANE.

● Φ_{Fh} SPHERICAL ANGULAR COORDINATE OF VEHICLE MEASURED NORMAL TO NOMINAL SUN TRAJECTORY PLANE.

Ψ_F SPHERICAL ANGULAR COORDINATE OF VEHICLE MEASURED IN NOMINAL EARTH TRAJECTORY PLANE.

Ψ_{FL} SPHERICAL ANGULAR COORDINATE OF VEHICLE MEASURED IN NOMINAL MOON TRAJECTORY PLANE.

Ψ_{Fh} SPHERICAL ANGULAR COORDINATE OF VEHICLE MEASURED IN NOMINAL SUN TRAJECTORY PLANE.

Φ_{ih} HELIOCENTRIC LATITUDE OF LAUNCH POINT.

Φ_{iL} SELENOCENTRIC LATITUDE OF LAUNCH POINT.
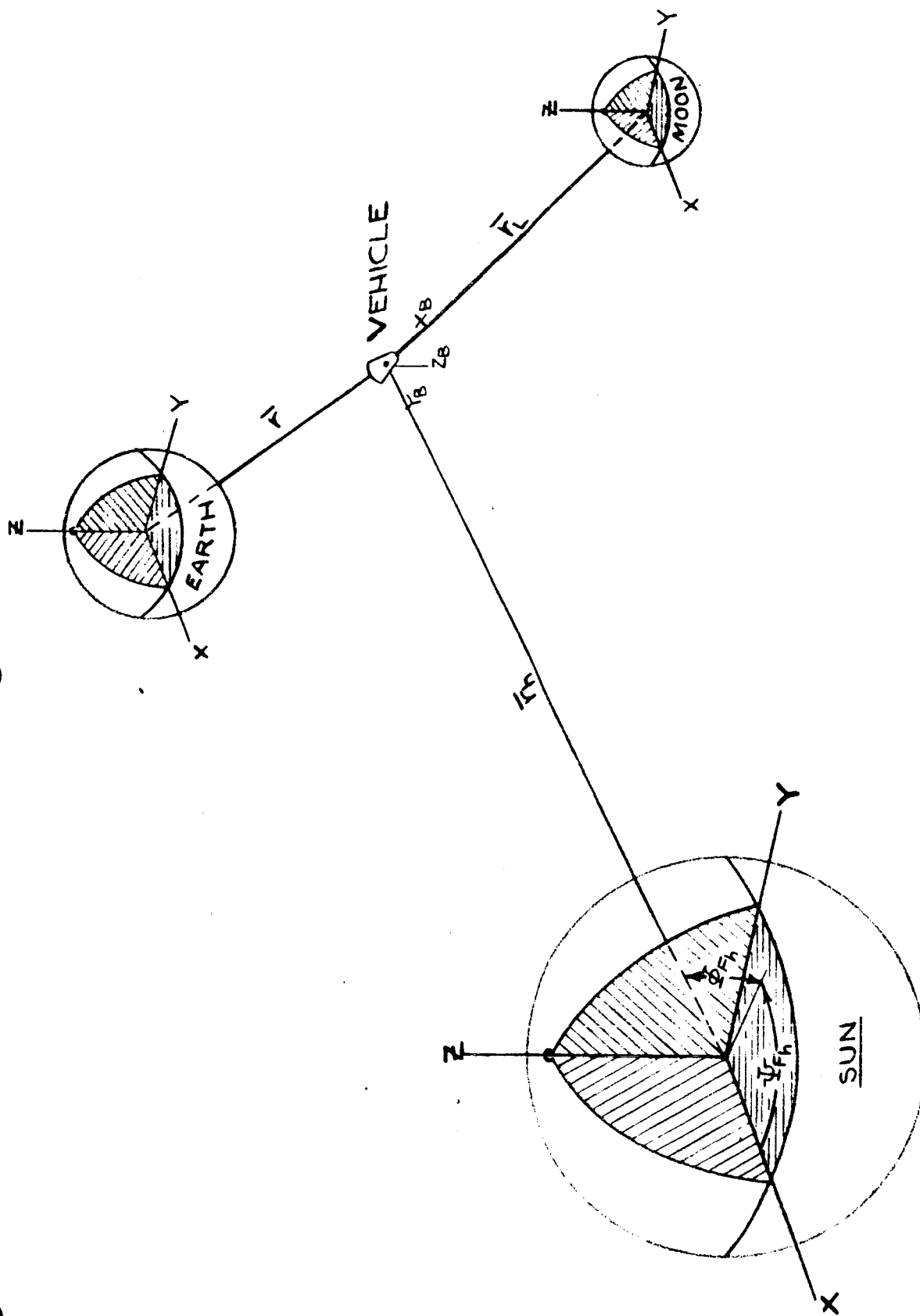
m VEHICLE MASS.

FIGURE I

GRAVITY VECTORS

THE VEHICLE TO EARTH GRAVITY VECTOR IS DESCRIBED IN PART I, MODEL TWO AS:

$$-\frac{K}{r^2}\,i_F \quad \text{WHERE} \quad K = g_e R_o^2, \quad K = 1.408 \times 10^{16}\,FT^3\,SEC^{-2} \quad (1)$$

SIMILARLY THE VEHICLE TO MOON VECTOR CAN BE DESCRIBED AS:

$$-\frac{K_L}{r_L^2}\,i_{F_L} \quad \text{WHERE} \quad K_L = g_L R_{o_L}^2, \quad K_L = 1.778 \times 10^{14}\,FT^3\,SEC^{-2} \quad (2)$$

IN THE SAME MANNER, THE VEHICLE TO SUN VECTOR CAN BE WRITTEN AS:

$$\frac{-K_h}{r_h^2}\,i_{F_h} \quad \text{WHERE} \quad K_h = g_h R_{o_h}^2, \quad K_h = 1.879 \times 10^{22}\,FT^3\,SEC^{-2} \quad (3)$$

RELATIVE TO THE "F" FRAME, THESE EQUATIONS ARE SUMMED UP TO FORM A FOUR-BODY GRAVITATIONAL EQUATION:

$$\frac{\bar{g}}{m} = \left[-\frac{K}{r^2} - \frac{K_L}{r_L^2} - \frac{K_h}{r_h^2}\right]i_F \qquad (4)$$

FOLLOWING THE ANALYSIS OF PAGES 30 & 31 OF PART I, MODEL TWO, THE MOON AND SUN REFERENCED TRANSLATION EQUATIONS OF ACCELERATION MAY BE WRITTEN AS:

$$\bar{a}_L = \left[\ddot{r}_L - r_L\left(\dot{\Phi}_{F_L}^2 + \dot{\Psi}_{F_L}^2 \cos^2\Phi_{F_L}\right)\right]i_{F_L} + \left[\left(2\dot{r}_L\dot{\Psi}_{F_L} + r_L\ddot{\Psi}_{F_L}\right)\cos\Phi_{F_L}\right.$$
$$\left. - 2r_L\dot{\Phi}_{F_L}\dot{\Psi}_{F_L}\sin\Phi_{F_L}\right]j_{F_L} + \left[r_L\ddot{\Phi}_{F_L} + 2\dot{r}_L\dot{\Phi}_{F_L} + r_L\dot{\Psi}_{F_L}^2\sin\Phi_{F_L}\cos\Phi_{F_L}\right]k_{F_L} \quad (5)$$

$$\bar{a}_h = \left[\ddot{r}_h - r_h\left(\dot{\Phi}_{F_h}^2 + \dot{\Psi}_{F_h}^2 \cos^2\Phi_{F_h}\right)\right]i_{F_h} + \left[\left(2\dot{r}_h\dot{\Psi}_{F_h} + r_h\ddot{\Psi}_{F_h}\right)\cos\Phi_{F_h}\right.$$
$$\left. - 2r_h\dot{\Phi}_{F_h}\dot{\Psi}_{F_h}\sin\Phi_{F_h}\right]j_{F_h} + \left[r_h\ddot{\Phi}_{F_h} + 2\dot{r}_h\dot{\Phi}_{F_h} + r_h\dot{\Psi}_{F_h}^2\sin\Phi_{F_h}\cos\Phi_{F_h}\right]k_{F_h} \quad (6)$$

COMBINING EQUATIONS (4), (5) AND (6) ABOVE WITH EQUATIONS (56), (57) AND (58) ON PAGE 33 OF PART I, MODEL TWO, AND DISREGARDING AERODYNAMIC AND EARTH OBLATENESS FORCES, THE MIDCOURSE TRANSLATIONAL EQUATIONS OF MOTION MAY NOW BE WRITTEN AS FOLLOWS:

$$\ddot{r} - r\left(\dot{\Phi}_F^2 + \dot{\Psi}_F^2 \cos^2\Phi_F\right) + \ddot{r}_L - r_L\left(\dot{\Phi}_{FL}^2 + \dot{\Psi}_{FL}^2 \cos^2\Phi_{FL}\right) + \ddot{r}_h - r_h\left(\dot{\Phi}_{F_h}^2 + \dot{\Psi}_{F_h}^2 \cos^2\Phi_{Fh}\right)$$

$$= -\frac{K}{r^2} - \frac{K_L}{r_L^2} - \frac{K_h}{r_h^2} + \frac{1}{m}\left(P_r + H_r\right) + \frac{1}{m}\left(P_{r_L} + H_{r_L}\right) + \frac{1}{m}\left(P_{r_h} + H_{r_h}\right) \tag{7}$$

$$\left(2\dot{r}\dot{\Psi}_F + r\ddot{\Psi}_F\right)\cos\Phi_F - 2r\dot{\Phi}_F\dot{\Psi}_F \sin\Phi_F + \left(2\dot{r}_L\dot{\Psi}_{FL} + r_L\ddot{\Psi}_{FL}\right)\cos\Phi_{FL}$$

$$-2r_L\dot{\Phi}_{FL}\dot{\Psi}_{FL}\sin\Phi_{FL} + \left(2\dot{r}_h\dot{\Psi}_{Fh} + r_h\ddot{\Psi}_{Fh}\right)\cos\Phi_{Fh} - 2r_h\dot{\Phi}_{Fh}\dot{\Psi}_{Fh}\sin\Phi_{Fh}$$

$$= \frac{1}{m}\left(P_{\Psi_F} + H_{\Psi_F}\right) + \frac{1}{m}\left(P_{\Psi_{FL}} + H_{\Psi_{FL}}\right) + \frac{1}{m}\left(P_{\Psi_{Fh}} + H_{\Psi_{Fh}}\right) \tag{8}$$

$$r\ddot{\Phi}_F + 2\dot{r}\dot{\Phi}_F + r\dot{\Psi}_F^2 \sin\Phi_F \cos\Phi_F + r_L\ddot{\Phi}_{FL} + 2\dot{r}_L\dot{\Phi}_{FL} + r_L\dot{\Psi}_{FL}^2 \sin\Phi_{FL}\cos\Phi_{FL}$$

$$+ r_h\ddot{\Phi}_{F_h} + 2\dot{r}_h\dot{\Phi}_{F_h} + r_h\dot{\Psi}_{Fh}^2 \sin\Phi_{Fh}\cos\Phi_{Fh} = \frac{1}{m}\left(P_{\Phi_F} + H_{\Phi_F}\right) + \frac{1}{m}\left(P_{\Phi_{FL}} + H_{\Phi_{FL}}\right)$$

$$+ \frac{1}{m}\left(P_{\Phi_{F_h}} + H_{\Phi_{Fh}}\right) \tag{9}$$

TO SIMPLIFY THE SOLUTION OF THESE EQUATIONS, THEY MAY BE SEPARATED INTO THREE SETS AS FOLLOWS:

$$\left.\begin{array}{l} \ddot{r} - r\left(\dot{\Phi}_F^2 + \dot{\Psi}_F^2 \cos^2\Phi_F\right) = -\frac{K}{r^2} + \frac{1}{m}\left(P_r + H_r\right) \\[2ex] \left(2\dot{r}\dot{\Psi}_F + r\ddot{\Psi}_F\right)\cos\Phi_F - 2r\dot{\Phi}_F\dot{\Psi}_F\sin\Phi_F = \frac{1}{m}\left(P_{\Psi_F} + H_{\Psi_F}\right) \\[2ex] r\ddot{\Phi}_F + 2\dot{r}\dot{\Phi}_F + r\dot{\Psi}_F^2 \sin\Phi_F\cos\Phi_F = \frac{1}{m}\left(P_{\Phi_F} + H_{\Phi_F}\right) \end{array}\right\} \tag{10}$$

$$\left.\begin{array}{l} \ddot{r}_L - r_L\left(\dot{\Phi}_{FL}^2 + \dot{\Psi}_{FL}^2 \cos^2\Phi_{FL}\right) = -\frac{K_L}{r_L^2} + \frac{1}{m}\left(P_{r_L} + H_{r_L}\right) \\[2ex] \left(2\dot{r}_L\dot{\Psi}_{FL} + r_L\ddot{\Psi}_{FL}\right)\cos\Phi_{FL} - 2r_L\dot{\Phi}_{FL}\dot{\Psi}_{FL}\sin\Phi_{FL} = \frac{1}{m}\left(P_{\Psi_{FL}} + H_{\Psi_{FL}}\right) \\[2ex] r_L\ddot{\Phi}_{FL} + 2\dot{r}_L\dot{\Phi}_{FL} + r_L\dot{\Psi}_{FL}^2 \sin\Phi_{FL}\cos\Phi_{FL} = \frac{1}{m}\left(P_{\Phi_{FL}} + H_{\Phi_{FL}}\right) \end{array}\right\} \tag{11}$$

$$\left.\begin{array}{l} \ddot{r}_h - r_h\left(\dot{\Phi}_{F_h}^2 + \dot{\Psi}_{F_h}^2 \cos^2\Phi_{Fh}\right) = -\frac{K_h}{r_h^2} + \frac{1}{m}\left(P_{r_h} + H_{r_h}\right) \\[2ex] \left(2\dot{r}_h\dot{\Psi}_{Fh} + r_h\ddot{\Psi}_{Fh}\right)\cos\Phi_{Fh} - 2r_h\dot{\Phi}_{Fh}\dot{\Psi}_{Fh}\sin\Phi_{Fh} = \frac{1}{m}\left(P_{\Psi_{Fh}} + H_{\Psi_{Fh}}\right) \\[2ex] r_h\ddot{\Phi}_{F_h} + 2\dot{r}_h\dot{\Phi}_{F_h} + r_h\dot{\Psi}_{Fh}^2 \sin\Phi_{Fh}\cos\Phi_{Fh} = \frac{1}{m}\left(P_{\Phi_{Fh}} + H_{\Phi_{Fh}}\right) \end{array}\right\} \tag{12}$$

3.

MATHEMATICAL MODEL FOR SPACE RENDEZVOUS

(REFERENCE MD 59-272)

LIST OF SYMBOLS

a — SEMI-MAJOR AXIS (FEET OR MILES)

A — BINORMAL VELOCITY COMPONENT (FT/SEC) FERRY VEHICLE

B — BINORMAL VELOCITY COMPONENT (FT/SEC) TARGET VEHICLE

C — CIRCUMFERENTIAL VELOCITY COMPONENT (FT/SEC)

e — ECCENTRICITY

f — TRUE ANOMALY $\theta - \omega$ (RADIANS)

$F_{x_1}, F_{y_1}, F_{z_1}$ — THRUST OR PERTURBATION ACCELERATIONS (HOMING) (FT/SEC^2)

H — VELOCITY THRUST COEFFICIENT

I — TOTAL IMPULSE FUNCTION (FT/SEC)

I_1 — IMPULSE AT DEPARTURE POINT (FT/SEC)

I_2 — IMPULSE AT ARRIVAL POINT (FT/SEC)

(i, j, k) — UNIT VECTORS IN A CIRCUMFERENTIAL, BINORMAL RADIAL SYSTEM

i — INCLINATION

h — ANGULAR MOMENTUM PER UNIT MASS (FT^2/SEC)

K — PROPORTIONAL THRUST COEFFICIENT

L — LEAD THRUST COEFFICIENT

n — RATE OF CHANGE OF MEAN ANOMALY (AVERAGE ANGULAR VELOCITY - RADIANS/SEC)

p — SEMI-LATUS RECTUM (FEET OR MILES)

R — RADIAL VELOCITY COMPONENT (FT/SEC)

r — RADIUS TO SATELLITE (FEET OR MILES)

S — LAPLACE TRANSFORM FREQUENCY VARIABLE

$X(s)$ — LAPLACE TRANSFORM COORDINATE AXIS

$Y(s)$ — LAPLACE TRANSFORM COORDINATE AXIS

$Z(s)$ — LAPLACE TRANSFORM COORDINATE AXIS

α_1 — ANGLE BETWEEN TRANSFER ORBIT PLANE AND INITIAL ORBIT PLANE (RADIANS)

α_2 — ANGLE BETWEEN TRANSFER ORBIT PLANE AND TERMINAL ORBIT PLANE (RADIANS)

θ — ANGLE FROM ASCENDING NODE TO POSITION IN TRANSFER ORBIT (RADIANS)

$\Delta\theta$ — $\theta_2 - \theta_1$ (RADIANS)

μ — GRAVITATIONAL CONSTANT (FT^3/SEC^2) 1.4072203×10^{16}

ϕ_1 — ANGLE FROM REFERENCE AXIS TO POSITION IN INITIAL ORBIT (RADIANS)

ϕ_2 — ANGLE FROM REFERENCE AXIS TO POSITION IN TERMINAL ORBIT (RADIANS)

ϕ_3 — $\theta_2 - \omega$ (RADIANS)

ω — ARGUMENT OF PERIGEE, ANGLE FROM REFERENCE AXIS TO PERIGEE POINT (RADIANS)

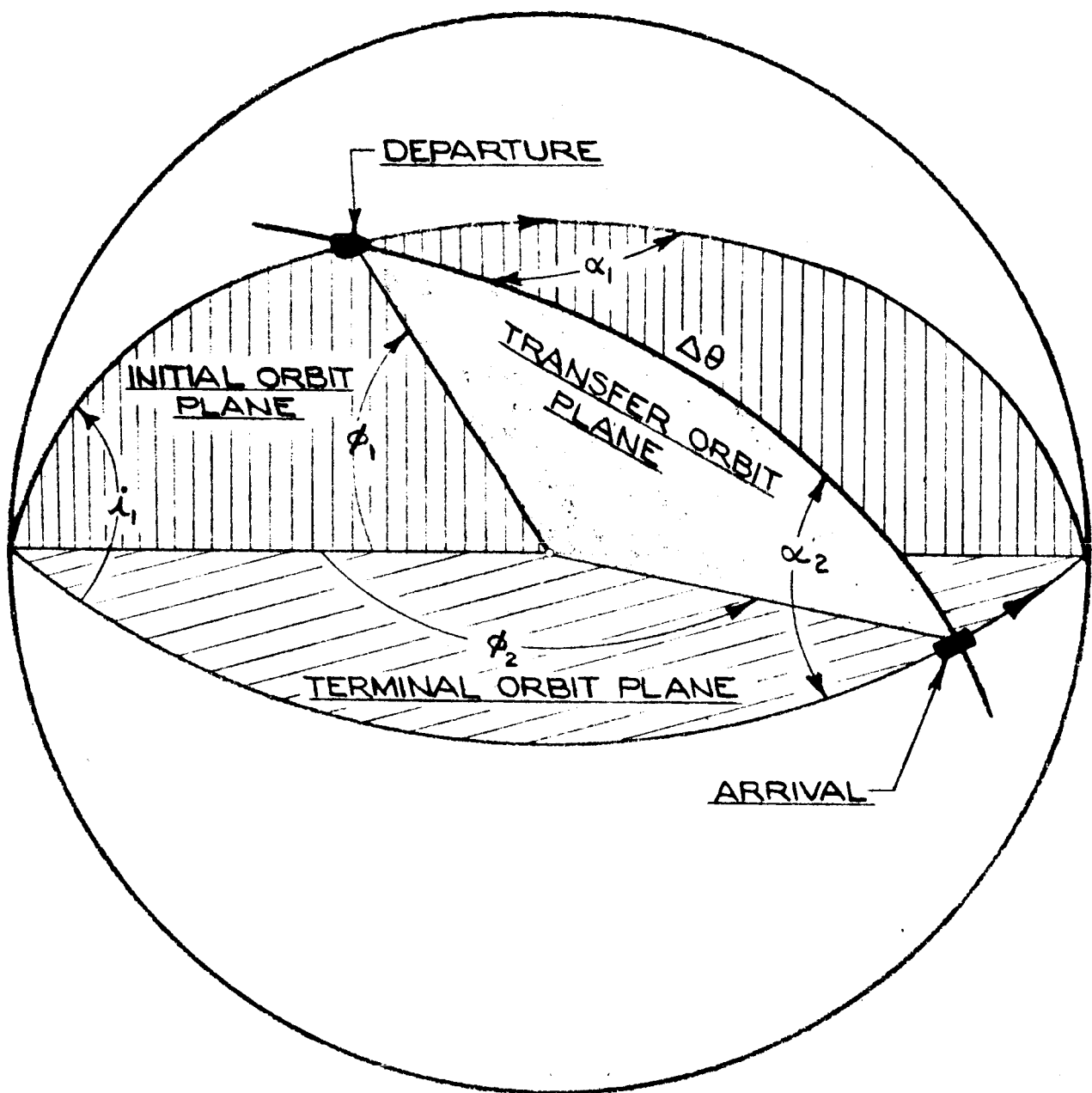Ω — RIGHT ASCENSION OF ASCENDING NODE (RADIANS)

SUBSCRIPTS:

1 — TRANSFER ORBIT AT DEPARTURE POINT

2 — TRANSFER ORBIT AT ARRIVAL POINT

3 — APPLIES TO ϕ_3 ONLY (SEE ABOVE)

11 — INITIAL ORBIT AT DEPARTURE POINT

22 — FINAL ORBIT AT ARRIVAL POINT

NOTE: WHEN APPLIED TO ORBITAL ELEMENTS $(p, e, \omega, i, \Omega)$ SUBSCRIPT 1 REFERS TO THE INITIAL ORBIT AND SUBSCRIPT 2 REFERS TO THE TERMINAL ORBIT. ORBITAL ELEMENTS OF TRANSFER ORBIT ARE DENOTED WITHOUT SUBSCRIPTS.

INITIAL ORBIT

TRANSFER ORBIT

ARRIVAL POINT

TARGET VEHICLE

TERMINAL ORBIT

LINE OF NODES

ϕ_2

ϕ_1

i_1

DEPARTURE POINT
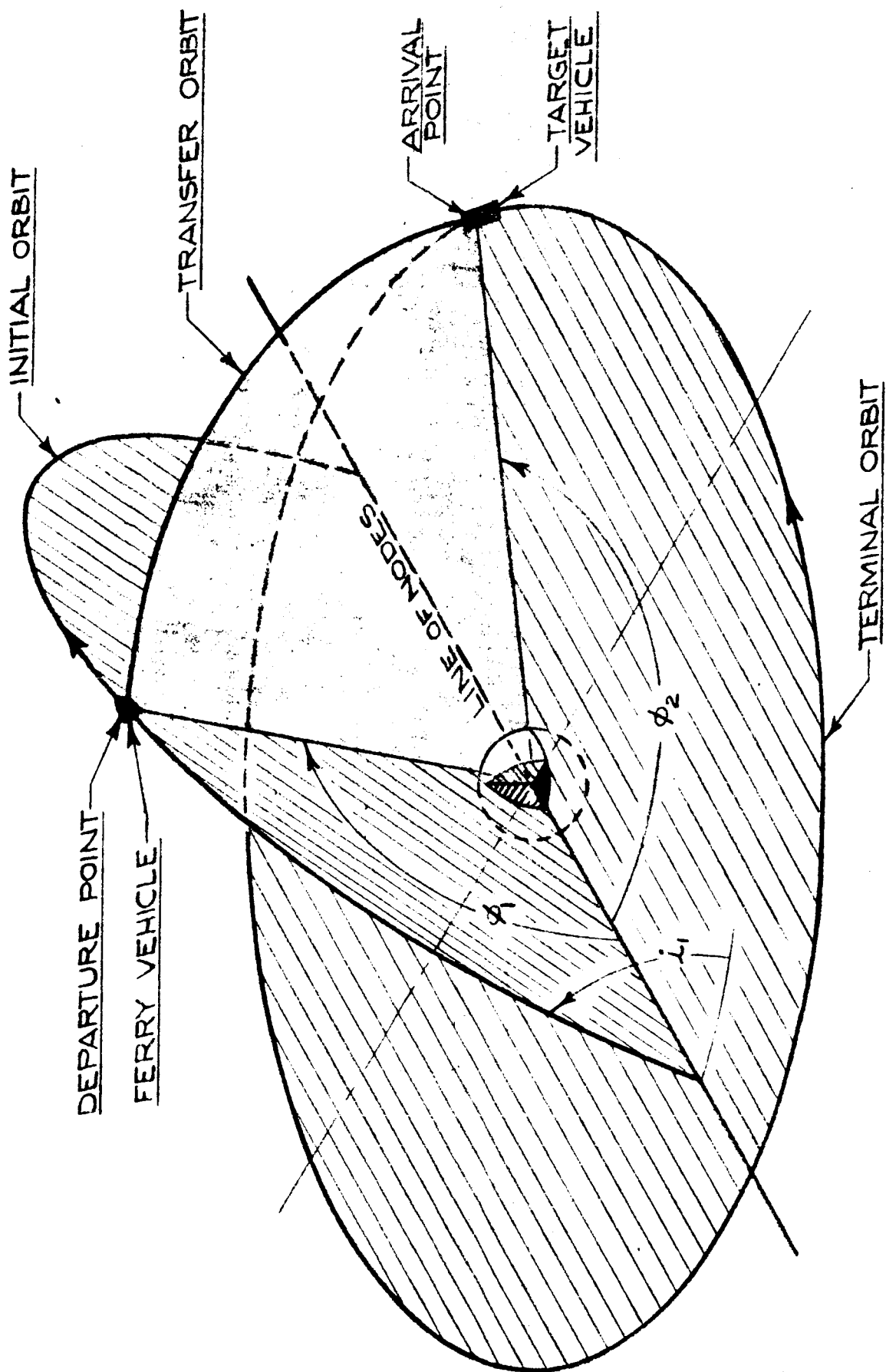
FERRY VEHICLE

TRANSFER GEOMETRY

FIGURE 1

1

PROJECTION ON UNIT SPHERE

FIGURE 2

SATELLITE RENDEZVOUS

INTRODUCTION:

SATELLITE RENDEZVOUS MAY BE CONSIDERED TO TAKE PLACE IN THREE PARTS. FIRST THE LAUNCH AND BOOST PHASE, IN WHICH THE FERRY VEHICLE IS PLACED IN AN ORBIT, AS NEARLY IDENTICAL AS POSSIBLE, TO THAT OF THE TARGET VEHICLE. SECONDLY, THE ORBITAL TRANSFER PHASE, IN WHICH THE FERRY VEHICLE PROPELS ITSELF INTO THE TRAJECTORY AND GENERAL VICINITY OF THE TARGET VEHICLE. THIRDLY, THE HOMING PHASE, IN WHICH THE FERRY VEHICLE IS GUIDED VISUALLY OR ELECTRONICALLY INTO DOCKING POSITION WITH THE APPROPRIATE ON BOARD PROPULSION SYSTEMS. THE EQUATIONS PRESENTED IN THIS REPORT WILL DEAL WITH THE TRANSFER AND HOMING PHASES OF THE SATELLITE RENDEZVOUS PROBLEM, BECAUSE OF THEIR APPLICATION TO THE APOLLO PROJECT.

ORBITAL TRANSFER

TO EFFECT AN ORBITAL TRANSFER REQUIRES A TWO STEP IMPULSE TO CHANGE THE FERRY VEHICLE FROM ITS INITIAL ORBIT WITH ELEMENTS p_1, e_1, ω_1, i_1 AND Ω_1, TO THE TARGET VEHICLE ORBIT WITH ELEMENTS p_2, e_2, ω_2, i_2 AND Ω_2. IF THE TERMINAL ORBIT IS USED AS A REFERENCE, $i_2 = 0$. ALSO, THE SYMMETRY OF THE CENTRAL GRAVITY FIELD MAKES THE ORIENTATION OF THE LINE OF NODES OF NO VALUE, THEREFOR $\Omega_1 = 0$. CONSEQUENTLY, SINCE THE VALUE OF Ω_2 IS ARBITRARY, THE PROBLEM IS SPECIFIED BY SEVEN PARAMETERS: AN INITIAL ORBIT WITH ELEMENTS p_1, e_1, ω_1, i_1, AND A TERMINAL ORBIT WITH ELEMENTS p_2, e_2, AND ω_2.

THE THREE PARAMETER FAMILY OF TRANSFERS BETWEEN TWO POINTS CAN BE DETERMINED USING THE RULE: GIVEN TWO POINTS NOT COLLINEAR WITH THE ORIGIN, AND A QUANTITY "p" GREATER THAN ZERO, THEIR EXISTS A UNIQUE CONIC PASSING THROUGH THE TWO POINTS, WITH THE ORIGIN AS FOCUS, AND "p" AS ITS SEMI-LATUS RECTUM.

LET $r_1 \theta_1$ AND $r_2 \theta_2$ BE THE TWO POINTS. IT IS NECESSARY TO OBTAIN $e \geq 0$ AND ω SUCH THAT:

$$r_1 = \frac{p}{1 + e\cos(\theta_1 - \omega)} \;;\qquad r_2 = \frac{p}{1 + e\cos(\theta_2 - \omega)} \tag{1}$$

AND:

$$\frac{p}{r_1} - 1 = e\cos(\theta_1 - \omega); \qquad \frac{p}{r_2} - 1 = e\cos(\theta_2 - \omega) \tag{2}$$

LET $\frac{p}{r_1} - 1 = A$ AND $\frac{p}{r_2} - 1 = B$,

THEN:

$$A = e\cos(\theta_1 - \omega); \qquad B = e\cos(\theta_2 - \omega) \tag{3}$$

OR:
$$\frac{A}{e} = \cos(\theta_1 - \omega); \qquad \frac{B}{e} = \cos(\theta_2 - \omega) \tag{4}$$

USING THE SPHERICAL TRIGONOMETRIC IDENTITY:

$$\sin(\theta_1 - \omega)\sin(\theta_2 - \theta_1) = \cos(\theta_1 - \omega)\cos(\theta_2 - \theta_1) - \cos(\theta_2 - \omega) \tag{5}$$

$$\sin(\theta_1 - \omega)\sin(\theta_2 - \theta_1) = \frac{A}{e}\cos(\theta_2 - \theta_1) - \frac{B}{e} \tag{6}$$

$$\sin(\theta_1 - \omega) = \frac{A\cos(\theta_2 - \theta_1) - B}{e\sin(\theta_2 - \theta_1)} \tag{7}$$

USING EQUATIONS (4) AND (7):

$$\sin^2(\theta_1 - \omega) + \cos^2(\theta_1 - \omega) = 1$$

$$\frac{A^2\cos^2(\theta_2 - \theta_1) - 2AB\cos(\theta_2 - \theta_1) + B^2}{e^2\sin^2(\theta_2 - \theta_1)} + \frac{A^2}{e^2} = 1$$

LET $\theta_2 - \theta_1 = \Delta\theta$ THEN:

$$\frac{A^2 - A^2\sin^2\Delta\theta - 2AB\cos\Delta\theta + B^2 + A^2\sin^2\Delta\theta}{\sin^2\Delta\theta} = e^2$$

$$e = \sqrt{\frac{A^2 - 2AB\cos\Delta\theta + B^2}{|\sin\Delta\theta|^2}} \tag{8}$$

EQUATIONS (4), (7), AND (8) FORM THE SOLUTION.

THE CONIC DEMONSTRATED BY THESE EQUATIONS IS UNIQUE, BUT AS AN ORBIT IT MAY BE TRAVERSED IN EITHER OF TWO DIRECTIONS. HOWEVER, WE SHALL ASSUME THAT THE PATH WHICH TRAVERSES AN ANGLE OF LESS THAN $180°$ SHALL BE SELECTED IN ALL CASES IN THAT IT IS THE SHORTEST ROUTE.

TRANSFER GEOMETRY

TRANSFER GEOMETRY CAN NOW BE DEFINED. GIVEN THE INITIAL AND TERMINAL ORBIT ELEMENTS $(p_1, e_1, \omega_1, i_1, p_2, e_2, \omega_2)$ A TRANSFER ORBIT IS PRESCRIBED BY SELECTING TWO ANGLES AND A DISTANCE. THESE ARE ϕ_1, THE POSITION OF THE DEPARTURE POINT ON THE INITIAL ORBIT, ϕ_2, THE POSITION OF THE ARRIVAL POINT ON THE TERMINAL ORBIT, AND p THE SEMI-LATUS RECTUM OF THE TRANSFER CONIC. THE CONIC IS UNDEFINED FOR VALUES OF $\phi_2 - \phi_1 = 180° = 0$.

IMPULSE FUNCTION

A DOUBLE-VALUED IMPULSE FUNCTION OF THE VARIABLES ϕ_1, ϕ_2 AND p MAY NOW BE DEFINED. THE RADIAL AND CIRCUMFERENTIAL VELOCITIES OF A CONIC ARE GIVEN BY:

$$R = \sqrt{\tfrac{\mu}{p}} \; e \, \sin(\theta - \omega) \tag{9}$$

$$C = \sqrt{\tfrac{\mu}{p}} \left[1 + e \cos(\theta - \omega) \right] \tag{10}$$

THE IMPULSE FUNCTION IS THEN DEFINED BY:

$$I = I_1 + I_2$$

OR:

$$I = \sqrt{(R_1 - R_{11})^2 + (C_1 - C_{11} \cos\alpha_1)^2 + (C_{11} \sin\alpha_1)^2} + \sqrt{(R_2 - R_{22})^2 + (C_2 - C_{22}\cos\alpha_2)^2 + (C_{22}\sin\alpha_2)^2} \tag{11}$$

THE VARIOUS RELATIONSHIPS IN THE TRANSFER GEOMETRY CAN BE SEEN BY REFERING TO FIGURE 1 AND 2.

THE PARTIAL DERIVATIVES OF THE IMPLICIT IMPULSE FUNCTION WITH RESPECT TO ϕ_1, ϕ_2, AND p MAY NOW BE WRITTEN AS:

$$\partial I / \partial \phi_1 = \frac{R_1 - R_{11}}{I_1 \sin \Delta\theta} \left[R_2 \cos\alpha_1 - R_{11} \cos\Delta\theta \sqrt{\frac{p}{p_1}} + \left(\sqrt{\frac{\mu}{p_1}} - C_{11} \right) \sin\Delta\theta \right]$$

$$+ \frac{C_1 - C_{11} \cos\alpha_1}{I_1} \left[-R_{11} \sqrt{\frac{p}{p_1}} \right] + \frac{C_{11} - C_1 \cos\alpha_1}{I_1} \left[-R_{11} \right]$$

$$+ \frac{C_1 C_{11}}{I_1 \sin^3\Delta\theta} \left[\sin^2 i_1 \sin^2 \phi_2 \cos\Delta\theta \right]$$

$$+ \frac{R_2 - R_{22}}{I_2 \sin\Delta\theta} \left[R_1 \cos\alpha_1 - R_{11} \sqrt{\frac{p}{p_1}} \right] + \frac{C_2 C_{22}}{I_2 \sin^3\Delta\theta} \left[\sin^2 i_1 \sin\phi_1 \sin\phi_2 \right] \quad (12)$$

$$\partial I / \partial \phi_2 = \frac{R_1 - R_{11}}{I_1 \sin\Delta\theta} \left[-R_2 \cos\alpha_2 + R_{22} \sqrt{\frac{p}{p_2}} \right] + \frac{C_1 C_{11}}{I_1 \sin^3\Delta\theta} \left[-\sin^2 i_1 \sin\phi_1 \sin\phi_2 \right]$$

$$+ \frac{R_2 - R_{22}}{I_2 \sin\Delta\theta} \left[-R_1 \cos\alpha_2 + R_{22} \cos\Delta\theta \sqrt{\frac{p}{p_2}} + \left(\sqrt{\frac{\mu}{p_2}} - C_{22} \right) \sin\Delta\theta \right]$$

$$+ \frac{C_2 - C_{22} \cos\alpha_2}{I_2} \left[-R_{22} \sqrt{\frac{p}{p_2}} \right] + \frac{C_{22} - C_2 \cos\alpha_2}{I_2} \left[-R_{22} \right]$$

$$+ \frac{C_2 C_{22}}{I_2 \sin^3\Delta\theta} \left[-\sin^2\phi_1 \sin^2 i_1 \cos\Delta\theta \right] \quad (13)$$

$$\partial I / \partial p = \frac{R_1 - R_{11}}{2 I_1 p \sin\Delta\theta} \left[R_1 \sin\Delta\theta - 2(1 - \cos\Delta\theta) \sqrt{\frac{\mu}{p}} \right]$$

$$+ \frac{C_1 - C_{11} \cos\alpha_1}{2 I_1 p} \left[C_1 \right] + \frac{R_2 - R_{22}}{2 I_2 p \sin\Delta\theta} \left[R_2 \sin\Delta\theta + 2(1 - \cos\Delta\theta) \sqrt{\frac{\mu}{p}} \right]$$

$$+ \frac{C_2 - C_{22} \cos\alpha_2}{2 I_2 p} \left[C_2 \right] \quad (14)$$

THUS, A CLASS OF ORBITAL TRANSFER EQUATIONS HAVE BEEN FORMULATED AND DEFINED.

HOMING

HOMING CAN BE CONSIDERED AS THE ACT OF GUIDING A CRAFT TO A SATELLITE TARGET WITH INITIAL POSITION AND VELOCITY DIFFERENCES OF APPROXIMATELY 20 MILES AND 100 FT/SEC RESPECTIVELY, TO WITHIN .05 MILES AND 2.0 FT/SEC. THUS, ORBIT TRANSFER ERRORS ARE CORRECTED UNTIL THE DOCKING MANEUVER CAN BE ACCOMPLISHED. A COORDINATE SYSTEM WITH THE ORIGIN AT THE TARGET VEHICLE IS ORIENTED SUCH THAT:

THE x_1-AXIS IS IN THE TARGET PLANE POINTING FORWARD

THE y-AXIS IS DIRECTED BINORMALLY

THE z_1-AXIS IS RADIAL

THEN, THE FIRST ORDER EQUATIONS OF MOTION ARE:

$$F_{x_1} = \ddot{x}_1 + \frac{\mu}{r^3}\left(1 - \frac{p}{r}\right)x_1 + \frac{2h}{r^2}\dot{z}_1 - \frac{2\mu e \sin f}{r^3}z_1$$

$$F_{y_1} = \ddot{y}_1 + \frac{\mu}{r^3}y_1$$

$$F_{z_1} = \ddot{z}_1 - \frac{\mu}{r^3}\left(2 + \frac{p}{r}\right)z_1 - \frac{2h}{r^2}\dot{x}_1 + \frac{2\mu e \sin f}{r^3}x_1 \qquad (15)$$

MAKING THE FOLLOWING SUBSTITUTIONS:

$$h = \sqrt{\mu p}; \quad p = a(1 - e^2); \quad n = \sqrt{\frac{\mu}{a^3}}; \quad r = \frac{p}{1 + e \cos f}$$

AND CONSIDERING THE PATH OF MOTION TO BE CIRCULAR BY LETTING $e \to 0$, EQUATION (15) BECOMES:

$$F_{x_1} = \ddot{x}_1 - e n^2 (x_1 \cos f + 2 z_1 \sin f) + 2n(1 + 2e \cos f)\dot{z}_1$$

$$F_{y_1} = \ddot{y}_1 + n^2(1 + 3e \cos f)y_1$$

$$F_{z_1} = \ddot{z}_1 - 2e n^2 x_1 \sin f - n^2(3 + 10e \cos f)z_1 - 2n(1 + 2e \cos f)\dot{x}_1 \qquad (16)$$

NOTE THAT THE "e" TERMS STILL REMAIN BUT THAT THE "e^2" AND "e^3" TERMS HAVE BEEN CANCELED IN AS MUCH AS THIS SIMPLIFIES THE EQUATIONS WITHOUT ANY APPRECIABLE LOSS IN ACCURACY. EQUATIONS (16) CAN BE FURTHER SIMPLIFIED IF THE "e" TERMS ARE REMOVED ENTIRELY. THIS HAS BEEN SHOWN TO BE PERMISSIBLE PROVIDED THAT RENDEZVOUS IS COMPLETED WITHIN ONE ORBIT OF THE VEHICLE.

THE EQUATIONS CAN THEN BE WRITTEN:

$$\underline{Fx_1 = \ddot{x}_1 + 2n\dot{z}_1}$$

$$\underline{Fy_1 = \ddot{y}_1 + n^2 y_1}$$

$$\underline{Fz_1 = \ddot{z}_1 - 3n^2 z_1 - 2n\dot{x}_1}$$

$\left.\right\}$ (17)

SINCE THESE EQUATIONS HAVE CONSTANT COEFFICIENTS THEY CAN BE WRITTEN IN TERMS OF LA PLACE TRANS- FORMS AS FOLLOWS:

$$X(s) = \frac{(\dot{x}_0 + 2nz_0)\left[s^2 + Hns + (K-3)n^2\right]}{D(s)} + \frac{x_0(s+Hn)\left[s^2 + Hns + (K-3)n^2\right]}{D(s)}$$

$$-\frac{(\dot{z}_0 - 2nx_0)(2ns + Ln^2)}{D(s)} - \frac{z_0(s+Hn)(2ns + Ln^2)}{D(s)} \qquad (18)$$

$$Y(s) = \frac{(s+Hn)y_0 + \dot{y}_0}{s^2 + Hns + (K+1)n^2} \qquad (19)$$

$$Z(s) = \frac{(\dot{z}_0 - 2nx_0)(s^2 + Hns + Kn^2)}{D(s)} + \frac{z_0(s+Hn)(s^2 + Hns + Kn^2)}{D(s)}$$

$$+\frac{(\dot{x}_0 + 2nz_0)(2ns + Ln^2)}{D(s)} + \frac{x_0(s+Hn)(2ns + Ln^2)}{D(s)} \qquad (20)$$

WHERE:

$$\underline{D(s) = s^4 + 2Hns^3 + (H^2 + 2K + 1)n^2 s^2 + \left[H(2K-3) + 4L\right]n^3 s + (K^2 - 3K + L^2)n^4}$$

INTEGRATION OF THE COMPLETE EQUATIONS OF MOTION FOR BOTH THE TARGET AND FERRY VEHICLE MUST BE PERFORMED FOR PURPOSES OF RENDEZVOUS.

STEERING:

TO ACCURATELY STEER THE VEHICLE TO THE POINT OF RENDEZVOUS A "LEAD" TERM PERPENDICULAR TO THE LINE OF SIGHT MUST BE INCLUDED IN THE STEERING EQUATIONS WHICH ARE COMPRISED OF THREE PARTS:

(1) A LINE OF SIGHT THRUST TOWARD THE TARGET;

$$-Kn^2(x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

(2) A "DAMPING" TERM PROPORTIONAL TO VELOCITY IN THE MOVING SYSTEM;

$$-Hn(\dot{x}_1\hat{i} + \dot{y}_1\hat{j} + \dot{z}_1\hat{k})$$

(3) A "LEAD" TERM PERPENDICULAR TO THE LINE OF SIGHT, AND DIRECTED WITH A SENSE OPPOSITE TO THAT OF THE VEHICLE'S MOTION ABOUT THE EARTH OR MOON;

$$Ln^2(x_1\hat{k} - z_1\hat{i})$$

THUS:

$$\left.\begin{aligned}
F_{x_1} &= -Kn^2 x_1 - Hn\dot{x}_1 - Ln^2 z_1\\
F_{y_1} &= -Kn^2 y_1 - Hn\dot{y}_1\\
F_{z_1} &= -Kn^2 z_1 - Hn\dot{z}_1 + Ln^2 x_1
\end{aligned}\right\} \quad (21)$$

ARE THE STEERING EQUATIONS FOR RENDEZVOUS.

CONCLUSION:

RENDEZVOUS, IN ITS TRANSFER AND HOMING PHASES, HAS BEEN DESCRIBED MATHEMATICALLY IN THIS REPORT. OF NECESSITY, TWO VEHICLES MUST BE IN ORBIT PRIOR TO THESE PHASES. THESE VEHICLES ARE REFERRED TO AS THE TARGET AND FERRY VEHICLES. THE FERRY VEHICLE'S TRAJECTORY AND LOCATION AT ANY TIME SHALL BE COMPUTED BY THE EARTH OR LUNAR ENVIRONMENT SECTIONS OF THE COMPUTER. HOWEVER, FOR PURPOSES OF TRAINING, THE TARGET VEHICLE'S TRAJECTORY AND LOCATION CAN BE PRE-PROGRAMMED OR TAPED INTO THE RENDEZVOUS PORTION OF THE COMPUTER. THEN, THE FERRY VEHICLE, USING ITS

ONBOARD PROPULSION AND REACTION CONTROL
CAPABILITY PERFORMS THE NECESSARY RENDEZVOUS
MANEUVERS.

BY PROGRAMMING AND COMPUTING SIMULATED
RENDEZVOUS PROBLEMS ON AN IBM 7090, THE OPTIMUM
VALUES FOR H, K, AND L IN THE STEERING EQUATIONS
WERE FOUND TO BE 6, 16, AND 8 RESPECTIVELY. OPTIMUM
IMPULSE QUANTITIES FOR PARTICULAR PROBLEMS WERE
ALSO ASCERTAINED.